XXIV. MATERIALS FAILURE THEORY, STRAIN ENERGY, PRINCIPAL STRESSES EIGENVALUE PROBLEM; HOW THESE THREE HISTORICAL AREAS ARE INTERRELATED AND MUTUALLY REINFORCING THUS ENABLING NEW RESULTS FOR MATERIALS FAILURE

1. Introduction

Consider the three disciplines or sub-disciplines of strain energy, the eigenvalue problem of principal stresses, and materials failure theory. What could these three seemingly individually self-contained areas have to do with each other? That is the subject of this entire paper.

First of all, all three topics were of classical origin yet all three are still of everyday, immediate relevance. Two of the three are complete, codified, absolutely beyond dispute or uncertainty, and universally employed. The third area is in a dreadful state of disrepair. Materials failure is unrecognizable as an organized discipline. A very brief historical summary of the three areas will set the stage for why the three of them are crucially inter-related and how this relationship can enable a considerable step toward rescuing the beleaguered third area.

Right from the beginning the great mathematician Cauchy [1] saw it all with incredible, almost unimaginable clarity. He immediately conceived the tensor valued character for stress and formulated the equilibrium equations with these. Navier was previously involved also. Because stress is tensor valued, with remarkable insight he, Cauchy, understood the role of coordinate transformations and recognized the existence of principal stresses and their special character. Cauchy viewed this through his stress quadric. The eigenvalue problem of the principal stresses arises directly from this.

Cauchy also recognized the geometric role of strains and the relationship between stresses and strains. The only thing he didn't quite get right was the exact form for the stress-strain relations. He and many others considered the relationship to be that of the attraction or repulsion between neighboring materials points, thus resulting in a one constant theory of elasticity. It was an eminently logical assumption at the time, but it was incorrect.

The one constant theory of elasticity didn't fit all the available testing data of the times and uncertainty and controversy over it ruled. A little later someone stepped forward with a further remarkable, almost earth shaking new insight. In one word it was energy. In two words, strain energy. George Green [2] was the man and the general concept of strain energy brought with it two materials constants. The disagreeable state of uncertainty and competition persisted and resisted resolution. Much later Love [3] gave a masterful and complete account of the one constant versus two constants controversy, and Timoshenko [4] gave a concise summary of these historical developments.

The time gap between Cauchy's foundational form for the theory (followed fairly shortly by Green's contribution) and the final agreement on the two constant form was an extremely lengthy 30 or more years. So two of the three areas of interest here have existed and been in practice for over 150 years. The theory of elasticity superficially seemed to be complete. On a deeper level though it couldn't possibly be considered to be complete. Fluids don't fail in any conventional sense of the term, but solids do fail, usually catastrophically. Solids cannot bear and support unlimited loads. There never will be completion of elasticity theory until the associated theory of elastic materials failure itself is synthesized, completed, and in practice.

It is a tremendous tribute to Coulomb [5] that right from the beginning of the new enlightenment he recognized the need for understanding and formalizing the treatment of materials failure. Each of the early technical pioneers had their own distinctive ideas on how to characterize failure, each hoping theirs to be the universal, general form. More than a hundred years later Mohr [6] picked up on Coulomb's valiant efforts and formulated the Coulomb-Mohr theory of materials failure, presented and presumed as a general failure criterion.

None of these efforts were successful. It was von Karman [7] who proved the incorrectness of the Coulomb-Mohr form. Another unsuccessful example was the much later Drucker-Prager form [8]. The history of materials failure investigation has been totally inconclusive and totally frustrating.

The only success was the remarkable development of fracture mechanics. But even that did not and does not supplant the need for a three dimensional theory of materials failure, one that would be the natural complement to the theory of elasticity. The widely used Mises [9] and also the Tresca criteria are completely incapable of treating general materials failure. Unfortunately that doesn't stop people from indiscriminatingly using them on everything. All of these matters were laid out and fully discussed by Christensen [10].

The work to be given here is directly responsive to the glaring inadequacy or absence of a scientific mathematical basis for treating materials failure. In particular, the classical eigenvalue problem for principal stresses and the classical formulation for strain energy will be used to develop the complementary and comprehensive theory of isotropic materials failure. The failure theory will be shown to be directly obtained from the other two disciplines, and to be of surprisingly simple form and of physical validity and realism.

2. The Eigenvalue Problem for the Principal Stresses

As one of the founding and sustaining pillars of the mechanics of materials, for the principal stresses and the principal directions to exist, the determinate of the stresses must vanish as in

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0 \tag{1}$$

leading to the characteristic equation for λ as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{2}$$

The coefficients in (2) are the stress invariants

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \tag{3}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \tag{4}$$

$$I_3 = \left|\sigma_{ij}\right| \tag{5}$$

and where the three roots of (2) are the three principal stresses

$$\lambda = \sigma_1, \quad \sigma_2, \quad \sigma_3 \tag{6}$$

The invariants (3)-(5) are invariant with respect to the orientation of the coordinate system used to describe the stress state. The first invariant I_1, Eq. (3), has a simple physical interpretation. Dividing Eq. (3) by 3 gives the mean normal stress. Mean normal stress causes a volume change. For this reason I_1 is sometimes termed as the dilatational invariant. This is in contrast to distortional behavior associated with shear stresses.

All stress states can be decomposed into combinations of dilatational and distortional stress states. The second invariant I_2 is neither a dilatational nor distortional stress state but it would be helpful to have I_2 expressed in terms that involve distortional stress states. To this end, the deviatoric stress tensor is introduced as

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk} \tag{7}$$

The first invariant of this deviatoric stress state vanishes thus it is independent of the dilatational stresses. The second invariant of (7) is given by

$$s_{ij}s_{ij}$$

and it is inherently distortional.

Any coefficient can be appended to this invariant form so long as it is consistently carried forward. Often the coefficient of 1/2 is used with yielding by the Mises criterion and the closely related concept of effective stress. But that is of no relevance or use here for general materials failure. For this latter general purpose, a coefficient of 3/2 is preferable and it will be used here. Define the second invariant of the deviatoric stresses as

$$J_2 = \frac{3}{2}s_{ij}s_{ij} \tag{8}$$

Writing this out in terms of components gives the very simple and direct form of

$$J_2 = \frac{1}{2}\left[\left(\sigma_{11}-\sigma_{22}\right)^2 + \left(\sigma_{22}-\sigma_{33}\right)^2 + \left(\sigma_{33}-\sigma_{11}\right)^2 + 6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)\right] \tag{9}$$

Invariant J_2 is always positive (as required for distortional states), I_2 is not. This will be of importance for the later strain energy considerations. Note that for uniaxial stress $J_2 = \sigma_{11}^2$ which will provide helpful calibration later.

Next find the relationship between the invariants J_2 and I_2, if such a relationship exists. It can be shown that

$$J_2 = I_1^2 - 3I_2 \tag{10}$$

With the use of (10) the eigenvalue problem characteristic equation (2) can be written in the alternate and apparently new mathematical form as

$$\lambda^3 - I_1\lambda^2 + \frac{1}{3}\left(I_1^2 - J_2\right)\lambda - I_3 = 0 \tag{11}$$

Eq. (11) probably provides the most fundamental statement of the principal stresses eigenvalue problem. It is completely specified by the four invariants

$$\begin{aligned} I_1 &= \sigma_{kk} \\ I_1^2 &= \left(\sigma_{kk}\right)^2 \\ J_2 &= \frac{3}{2}s_{ij}s_{ij} \\ I_3 &= \left|\sigma_{ij}\right| \end{aligned} \tag{12}$$

The two invariants I_1 and I_1^2 are more than just simple variations of each other. Both are independently required in this eigenvalue problem of the principal stresses. Each has separate status. For example, I_1^3 is also an invariant but it is of no significance for the eigenvalue problem. While the principal stresses and their associated principal directions are important, the guiding invariants (12) are even more important. This will be shown in the further developments of this paper.

It is of further interest to observe that in contrast to the I_2 situation, I_3 in (11) cannot be expressed in terms of I_1 and J_2 through using I_1^3 and I_1J_2. This proves that physical behavior in solids does occur beyond that which is directly predictable from only distortional and dilatational stress states. This will turn out to be of high importance in the treatment of materials failure.

The classical treatment of principal stresses with the attendant eigenvalue problem is the founding principle for the entire field of mechanics of materials. This eigenvalue problem for principal stresses (11) and the invariants (12) provides the basics tools for all further developments to be given here. It is independent of any particular materials symmetry.

The particular symmetry case of materials isotropy also has the same coordinate invariance for its properties and behaviors as the principal stress problem. It follows that it is very likely that the related physical properties for isotropic materials will require the use of the four invariants in (12) or a subset of them. Special interest here is in the behavior of strain energy and the failure for isotropic materials and how these relate to the eigenvalue problem of the principal stresses.

3. The Strain Energy for Isotropic Materials

Now, taking attention to isotropic materials the classical strain energy will be formulated. By common convention the term strain energy is used whether the energy is expressed in terms of elastic strains or elastic stresses. The interest here is in the stress case.

The strain energy must be of a positive definite form. It will be expressed in terms of the invariants in (12). The only terms in (12) that comply with the positive definite requirement are the middle two terms in (12), thus

$$U = \frac{\alpha I_1^2}{k} + \frac{\beta J_2}{\mu} \tag{13}$$

Eq. (13) necessarily associates the bulk modulus with the dilatational invariant term and the shear modulus with the distortional invariant term. Nondimensional parameters α and β are to be evaluated by the usual methods. The invariant I_2 is of no relevance and could not be directly used here.

It is found that the strain energy (13) is given by

$$U=\frac{1}{6}\left(\frac{I_1^2}{3k}+\frac{J_2}{\mu}\right) \tag{14}$$

Relation (14) will be converted to the alternative form involving the elastic modulus and Poisson's ratio through

$$\begin{aligned} \mu &= \frac{E}{2(1+\nu)} \\ k &= \frac{E}{3(1-2\nu)} \end{aligned} \tag{15}$$

Then finally for this form

$$U=\frac{1}{3E}\left[\frac{(1-2\nu)}{2}I_1^2+(1+\nu)J_2\right] \tag{16}$$

where I_1 is given by (3) and J_2 by (9). Relation (16) is the benchmark strain energy form.

Relations (14) and (16) are vastly simpler and more meaningful than those usually found for the strain energy of isotropic materials in the standard textbooks. More important than that, the strain energy form (16) will next be shown to provide valuable guidance on how to proceed with the closely related forms for the associated materials failure theory.

4. Isotropic Materials Failure Theory

The eigenvalue problem of the principal stresses is fundamental. As a consequence the failure of an isotropic material must be expressed in terms of the four invariants in (12) or a subset of them. It is likely that all four of them are involved with failure, as will be shown here.

Failure generally represents the termination of the elastic range, whether strain hardening is involved or not. This is the overriding requirement needed to proceed further.

The obvious candidate failure criterion would be a limit on the strain energy. But that could not be correct because it would require that materials would fail in hydrostatic compression. That certainly is not true, at least not within normal engineering limits.

We are motivated to look for some quantity, call it the failure potential, that is related to the strain energy but is independent of it. Furthermore, it must depend upon purely dilatational and distortional stress states, just as the strain energy does. Since failure is here interpreted to be the cessation of the range of elastic behavior, the failure potential will be expected to be much like the strain energy (16) and it will involve two constants. The J_2 term in (16) is acceptable but the I_1^2 term is not since that would just revert back to energy as the failure decider. From the list of invariants in (12) the only dilatational alternative to I_1^2 would be I_1. Thus we must take the failure potential as having the form

$$\alpha I_1 + \beta J_2 \leq 1 \tag{17}$$

where these new α and β's are to be evaluated in terms of common, fundamental and accessible failure properties.

Since the invariants in (17) arise through the eigenvalue problem of the principal stresses, it seems likely that α and β in (17) will be calibrated by strengths in the form of principal stresses. That is exactly what happens. The customary shear strength has no role to play here in the calibration process, nor should it. The widely used effective stress terminology is motivated by shear stress failure but that is a misguided concept that could be misleading here and it will not be employed.

The resulting forms for α and β in terms of the uniaxial tensile and compressive strengths, T and C, then give the completed failure potential as:

Failure Potential (Polynomial Invariants)

$$\left(\frac{1}{T}-\frac{1}{C}\right)I_1+\frac{1}{TC}J_2\leq 1 \qquad (18)$$

I_1 is the dilatational stress invariant (3) and J_2 is the distortional stress invariant (8) and (9). The failure result (18) is extraordinarily clear, concise, and compact. This form is the same as that derived in Ref. [10] by a very different method, there named the polynomial invariants method.

For T=C (18) becomes the Mises criterion, [9]. Otherwise there is an interaction between the two invariants in (18) that produces the means and modes of failure. More of this will be said at the end of this section. Distortion and dilatation explicitly control everything in (18).

Compare the strain energy (16) and the failure potential (18). Both involve J_2 but the strain energy includes I_1^2 whereas the failure potential involves I_1. Thus the strain energy and the failure potential have an intimate (but independent) relationship. They could be said to be duals, both come from the invariants (12) of the eigenvalue problem, both involve only pure states of dilatation and distortion, and finally both (16) and (18) are completely expressed (calibrated) in terms of only the uniaxial stress properties of strength.

Does the failure form (18) provide a mathematically and physically complete description of all possible failure modes for isotropic materials? The failure form (18) only uses two of the four fundamental invariants (12) that follow from the eigenvalue problem. That certainly raises some doubts about the completeness of (18). Those doubts are well founded. Where is a fracture mode of failure? It is not even implicitly involved in (18). Fracture must also somehow explicitly be involved and included.

In the present context, a widely acknowledged mode of fracture is given by

$$\sigma_1 \leq T$$

where temporarily σ_1 is taken to be the largest principal stress. However a failure criterion of this type has a major problem. It would cut off part of the Mises criterion in the 1st quadrant of a biaxial stress state. That would be unacceptable for very ductile materials.

Physical intuition suggests that the fracture type criterion shown above would only apply for the more brittle range of materials but not for the more ductile type of materials. There needs to be a specified cut off that separates the ductile and brittle materials classes. The materials type is specified by the value of T/C. The correct specification for the above fracture criterion is given by:

Fracture Criterion

$$\begin{gathered} For \quad 0 \leq \frac{T}{C} \leq \frac{1}{2} \\ \sigma_1 \leq T \\ \sigma_2 \leq T \\ \sigma_3 \leq T \end{gathered} \tag{19}$$

where these are the three principal stresses, in any order. If the cutoff value were taken as anything other than T/C=1/2 in (19) then there would be a step function discontinuity between the failure potential envelope (18) and the beginning of the fracture controlled envelope (19), as T/C is varied. Such a jump behavior would be physically inadmissible. This unacceptable behavior is most easily seen in two dimensional biaxial stress conditions, although it certainly covers all conditions. The fracture cutoff value at T/C=1/2 is of major significance. It provides the most basic and elementary natural division into ductile versus brittle materials.

The two failure criteria (18) and (19) are complete and they are competitive in operation. In any condition the more restrictive one controls the failure behavior. The failure potential and the fracture form are independent requirements. One without the other is not just incomplete, it would be incorrect. But both together form a complete and comprehensive failure theory.

Since (19) involves the principal stresses it is seen that all four invariants in (12) for the eigenvalue problem are involved in (19). This is a further intimate connection between all three of these basic disciplines.

It is advantageous to rewrite these controlling failure forms using nondimensional terms. Take

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{C} \tag{20}$$

Then (18) and (19) become

$$\begin{gathered} For \quad 0 \le \frac{T}{C} \le 1 \\ \left(1 - \frac{T}{C}\right)\left(\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33}\right) \\ + \frac{1}{2}\left[\left(\hat{\sigma}_{11} - \hat{\sigma}_{22}\right)^2 + \left(\hat{\sigma}_{22} - \hat{\sigma}_{33}\right)^2 + \left(\hat{\sigma}_{33} - \hat{\sigma}_{11}\right)^2 + 6\left(\hat{\sigma}_{12}^2 + \hat{\sigma}_{23}^2 + \hat{\sigma}_{31}^2\right)\right] \le \frac{T}{C} \end{gathered} \tag{21}$$

$$\begin{gathered} For \quad 0 \le \frac{T}{C} \le \frac{1}{2} \\ \hat{\sigma}_1 \le \frac{T}{C} \\ \hat{\sigma}_2 \le \frac{T}{C} \\ \hat{\sigma}_3 \le \frac{T}{C} \end{gathered} \tag{22}$$

Relation (21) can be written in the alternate form

$$\left(1 - \frac{T}{C}\right)\hat{\sigma}_{kk} + \left[\hat{\sigma}_{11}^2 + \hat{\sigma}_{22}^2 + \hat{\sigma}_{33}^2 - \hat{\sigma}_{11}\hat{\sigma}_{22} - \hat{\sigma}_{22}\hat{\sigma}_{33} - \hat{\sigma}_{33}\hat{\sigma}_{11} + 3\left(\hat{\sigma}_{12}^2 + \hat{\sigma}_{23}^2 + \hat{\sigma}_{31}^2\right)\right] \le \frac{T}{C} \tag{23}$$

The three areas of investigation here, the eigenvalue problem of the principal stresses, the strain energy, and the failure theory are now seen to be completely intertwined and mutually reinforcing in their validity. The eigenvalue problem is the basic, all encompassing form leading to the other two areas through its invariants. The strain energy and the failure potential are duals in their formation and function. The completion of the failure theory by the fracture criterion follows directly from the full form of the eigenvalue problem. It all is a complete and consistent formalism.

At the most basic level, the present approach for developing failure criteria from the principal stress eigenvalue problem corroborates the drastically different approach followed in Ref. [10]. Taken together they coordinate to form an unshakable foundation for this general theory of materials failure. The validity and viability of this failure theory was established in Ref. [10].

The failure mechanisms embedded in the failure theory are those of shear bands, voids nucleation, and maximum principal stress fracture. The first two failure mechanisms are implicit within the failure potential (18), voids nucleation through I_1 and shear bands through J_2. The third one, fracture, is explicitly stated by (19).

In sum, the principal stresses eigenvalue problem (11) and (12), the strain energy (16), and the failure theory (18) and (19) form a unified and cohesive treatment of the three disciplines. This unusual confluence of the three very diverse physical behaviors brings each area as well as the whole into clear and strong perspective. The first two areas are classical results and the third is a surprising and powerful consequence of them. They were not understood and recognized as a single formalism until now. This concludes the incredibly long gestation period for the third area. After 200+ years of searching it is now ready and primed for deployment. Failure theory has finally arrived.

5. Failure Due to Shear Stress With and Without Superimposed Hydrostatic Stress

The failure potential (18) is based upon dilatational and distortional states of stress and their interaction. In terms of applications of this failure theory there could be simple combinations of the two states of pure dilatation and pure distortion. The simplest case of this type is that of the direct interaction of a shear stress state and a hydrostatic (positive or negative) stress state. Begin with the shear stress by itself.

Failure Due to Shear Stress Only

Take the shear stress at failure as given by S. The failure potential (21) gives the failure as

$$\hat{S} \le \sqrt{\frac{1}{3}\left(\frac{T}{C}\right)} \tag{24}$$

The competitive fracture failure from (22) is

$$\hat{S} \leq \frac{T}{C} \tag{25}$$

These then give the complete failure specification as

$$For \quad 0 \leq \frac{T}{C} \leq \frac{1}{3} \qquad \hat{S} = \frac{T}{C} \tag{26}$$

$$For \quad \frac{1}{3} \leq \frac{T}{C} \leq 1 \qquad \hat{S} = \sqrt{\frac{1}{3}\left(\frac{T}{C}\right)} \tag{27}$$

Failure relations (26) and (27) are shown in Fig. 1

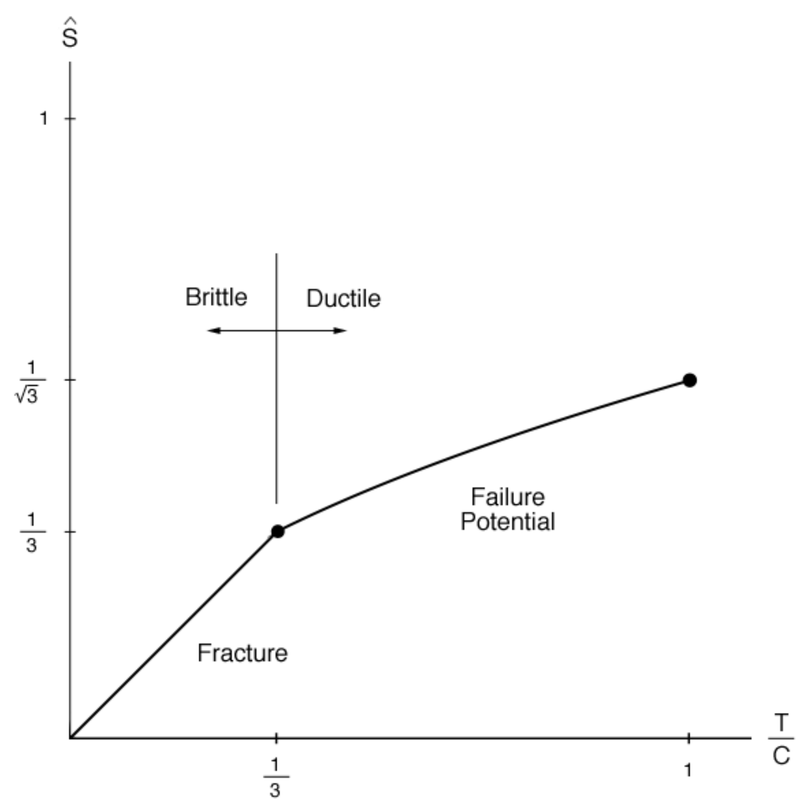

Fig. 1 Shear stress failure

It is seen from Fig. 1 that both the failure potential and the fracture criterion have distinct and vital functions. The failure potential by itself would have a physically impossible infinite slope at the origin and the fracture form by itself would far overestimate the failure at T/C=1, the Mises material case. Both competitive failure criteria are essential. Furthermore, the transition from brittle failure to ductile failure naturally occurs at T/C=1/3 in Fig. 1, for this particular stress state. The corresponding ductile versus brittle division for simple tension is at T/C=1/2.

<u>Failure Due to Shear Stress Plus Pressure</u>

Now take the case of an imposed shear stress plus a superimposed pressure p. The failure potential (21) gives the shear stress at failure S as

$$\hat{S} \leq \sqrt{\frac{1}{3}\left(\frac{T}{C}\right)+\left(1-\frac{T}{C}\right)\hat{p}} \tag{28}$$

while the fracture form (22) gives

$$\hat{S} \leq \frac{T}{C}+\hat{p} \tag{29}$$

Then the failure stresses are given by

$$\begin{aligned} &For \quad \frac{T}{C} \geq \frac{1}{2} \qquad && \hat{S}=(28) \\ &For \quad \frac{T}{C} \leq \frac{1}{2} \qquad && \hat{S}=(28)\, or\, (29)\, whichever\ is\ less \end{aligned} \tag{30}$$

The combined strength behaviors of shear stress plus pressure is given in Table 1.

$\hat{S}$ Shear Stress at Failure

	$\frac{T}{C}=0$	$\frac{T}{C}=\frac{1}{3}$	$\frac{T}{C}=\frac{1}{2}$	$\frac{T}{C}=\frac{2}{3}$	$\frac{T}{C}=1$
$\hat{p}=0$	0	1/3	0.408	0.471	0.577
$\hat{p}=\frac{1}{3}$	1/3	0.577	0.577	0.577	0.577
$\hat{p}=\frac{1}{2}$	1/2	2/3	0.646	0.624	0.577
$\hat{p}=\frac{2}{3}$	2/3	0.745	0.707	2/3	0.577
$\hat{p}=1$	1	0.882	0.816	0. 745	0.577

Table 1 Shear failure stress $\hat{S}$=S/C at superimposed pressure $\hat{p}$= p/C

From Table 1 it is seen that for a given T/C value, increasing pressure always increases the shear strength except at T/C=1 where the strength is independent of pressure. But for a given value of pressure increasing the T/C values increases the shear strength at low pressures, but decreases the shear strength at higher pressures. This may at first seem to be counter-intuitive but on further thought it is required behavior, most easily viewed through this general failure theory.

The general conclusion is that pressure enhances the shear strength and Table 1 shows the quantitative size of the effect. For the materials type T/C=1/2 the pressure effect at p=C exactly doubles the shear strength at no pressure. The pressure effect on the strength that is shown here is one of the most fundamental physical effects inherent in materials failure behavior.

Failure Due to Shear Stress Plus Positive Hydrostatic Stress

Finally consider the case of shear stress with a positive state of hydrostatic stress being superimposed. This has a very different behavior from the pressure case just examined.

Take the hydrostatic tension as σ. The failure criteria can be obtained from (28)-(30) with the pressure replaced by p= -σ. Consider three cases, all at T/C=2/3 but with $\hat{\sigma}$=0, $\hat{\sigma}$=1/3, and $\hat{\sigma}$=2/3. Using (28) it is found that

$$\begin{aligned}
&For \quad \frac{T}{C}=\frac{2}{3}, \quad \hat{\sigma}=0 && \hat{S}=\frac{1}{\sqrt{3}}=0.577 \\
&For \quad \frac{T}{C}=\frac{2}{3}, \quad \hat{\sigma}=\frac{1}{3} && \hat{S}=\frac{1}{3} \\
&For \quad \frac{T}{C}=\frac{2}{3}, \quad \hat{\sigma}=\frac{2}{3} && \hat{S}=0
\end{aligned} \tag{31}$$

With no hydrostatic tensile stress the shear strength is the largest. At $\hat{\sigma}$=2/3 the shear strength vanishes and the material actually fails under the tensile hydrostatic stress state by itself.

Many other illuminating examples of failure are given in Ref. [10], but those given here are not only the simplest, they are the most basic and revealing examples of isotropic materials failure.

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