



Exploration of Ductile, Brittle Failure Characteristics Through a Two Parameter Yield/Failure Criterion

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Abstract

A criterion is developed for determining whether a particular mode of failure is expected to be of ductile type or brittle type depending upon both the material type and the particular state of stressing the isotropic material to failure. The material type is specified by an α index obtained from two specific failure properties and a recently formulated failure theory. The ductile versus brittle criterion then involves the α material specification and the mean normal stress part of the imposed stress state. Several illustrative examples are given for different stress states and a spectrum of materials types. Closely related to the failure mode types are the orientations of the associated failure surfaces. In uniaxial tension, the present method shows a distinct failure orientation transition from ductile type to brittle type as the type of material changes.

Key Words

Yield, Failure, Ductile, Brittle, Ductile-Brittle Transition

Introduction and Background

For some purposes it is not sufficient to simply view a particular material as being either ductile or brittle in any and all possible conditions. Behaviors of these types also depend upon the particular state of stress under which the material is taken to failure. For example, superimposed pressure can convert what is commonly thought of as a brittle material into a ductile material for a particular stress state taken to failure. The explicit state of stress is a decisive factor. A stress state based failure criterion that distinguishes fracture from plastic flow could provide a step toward a more discriminating approach to the ductile versus brittle failure judgment.

Failure criteria for materials are widely employed but have been somewhat controversial. There are only two classical forms, the two-parameter Coulomb-Mohr form for general applications and the one parameter Mises form for ductile metals. The Tresca form can be considered to be a special case of the Coulomb-Mohr form. Three or more parameter forms generally have evolved simply as a means of having flexibility in fitting data. The advantages of a two-property failure form are apparent and the present work is concerned with a completely different type of two parameter (property) theory of failure for isotropic materials, as an alternative to both the two parameter Coulomb-Mohr form and the many multi-parameter empirical forms.

This recently developed failure theory has been given by Christensen [1], also giving references to earlier work and a brief historical account for the field. The treatment admits the Mises form for ductile materials at one extreme and a brittle, limiting case behavior at the other extreme. The present work is entirely concerned with developing a method for determining which type of failure mode (D or B) is likely to occur based upon both the material type and the particular state of stressing the material to failure. Neither specification by itself is sufficient to determine the expected D-B nature of a specific failure mode. The material type will be quantified through the new failure criteria.

This new criterion for homogeneous, isotropic material failure, Christensen [1], was derived in the tensor form given below

$$\alpha \left(\frac{\sigma_{ii}}{\kappa} \right) + \frac{3}{2} (1 + \alpha) \left(\frac{s_{ij}}{\kappa} \right) \left(\frac{s_{ij}}{\kappa} \right) \leq 1 \quad (1)$$

and

$$\sigma_1 \leq \frac{\kappa}{1 + \alpha} \quad \text{if } \alpha > 1 \quad (2)$$

where

$$\left. \begin{aligned} \kappa &= C \\ \alpha &= \frac{C}{T} - 1 \quad (0 \leq \alpha \leq \infty) \end{aligned} \right\} \quad (3)$$

where C and T are the uniaxial compressive and tensile strengths and where s_{ij} is the deviatoric stress tensor, and σ_1 is the algebraically largest principal stress. Thus, the failure theory is fully characterized by the uniaxial tensile and compressive stress values T & C. Criterion (2) is a fracture cutoff effect, which places further limits on (1). Examples will be given later. The details of the derivation of (1)-(3) are given in the reference cited above.

It is important to recognize the significance of the fracture criterion (2). Although maximum principal stress has often been used as a fracture criterion, it is also necessarily specified in (2) that it only applies in the brittle range of behavior, given by $\alpha > 1$. This latter restriction is of fundamental importance, and provides necessary coordination with the other physical effects implicit in the yield/strength criterion (1).

It also is important to differentiate the present approach from other widely recognized approaches. Usually the D-B transition has been considered in the context of dislocations emissions from crack tips. Whether or not a dislocation flow can cause shielding or blunting then is used to relate to the D-B criterion. For a prime example of this well motivated approach, see Rice and Thomson [2]. For a further example, Hirsch and Roberts [3] consider some of the complexities in modeling plastic zones and the D-B transition in metals. In the present work the interests are with much broader classes of materials than just those controlled by dislocations. In polymers for example there are virtually no useful criteria to be used to study D-B behavior. For this reason the macroscopic criteria (1)-(3) will be used which may apply across broad materials classes.

In complete component form (1) and (2) become

$$\begin{aligned}
& \left(\frac{1}{T} - \frac{1}{C} \right) (\sigma_{11} + \sigma_{22} + \sigma_{33}) \\
& + \frac{1}{TC} \left\{ \frac{1}{2} \left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] \right. \\
& \left. + 3(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \right\} \leq 1
\end{aligned} \tag{4}$$

$$\sigma_i \leq T \text{ if } \frac{T}{C} \leq \frac{1}{2} \tag{5}$$

The failure criteria (4) and (5) were extensively and favorably compared with data from many of these material types, Christensen [1]. Specifically, rather complete failure data sets for polypropylene, iron, a graphitic material and a mineral were compared with predictions from (4) and (5) and found to be much better than those from Coulomb-Mohr, and generally satisfactory. Specific materials examples of ductile and brittle behavior were given by Christensen [4,5].

It should be mentioned that the material classes intended to be covered by this criteria are those for $0 \leq \alpha \leq \infty$ which includes virtually all homogeneous, isotropic materials. Material types which are excluded are those with a significant amount of porosity which typically have $T/C > 1$ giving negative α .

In principal stress space the criterion (4) represents a paraboloid, inclined at equal angles to all three principal axes and opening up in the negative coordinate directions. The fracture cut off criterion (5) involves three planes normal to the principal stress axes which then intersect the paraboloid in cases of brittle materials with $T/C < 1/2$. This rather intricate and non-intuitive failure surface still devolves from only two failure parameters. In contrast, the two parameter Coulomb-Mohr form is simply a six sided pyramid in principal stress space, Paul [6]. Even more simple is the conical failure surface proposed by Drucker and Prager [7], also involving two parameters. With the two parameters adjusted to uniaxial tensile and compressive strengths, both the Coulomb-Mohr and Drucker-Prager forms strongly overestimate strengths in the tensile region of their apexes as well as in the opposite very compressive regions of their failure surfaces, Paul [6]. In a state of equal tri-axial tension, the Coulomb-Mohr method predicts a failure stress three times larger than that from (1) and (2), for all values of α . Some three

parameter failure forms may be seen in Paul [6] and in Jaeger and Cook [8]. The specific paraboloid form (4) has had some discussion and a few applications, see for example Raghava, Caddell and Yeh [9]. The present complete set of criteria (1)-(5) are of recent origin and have no historical precedent.

A Ductile versus Brittle Criterion Determined by the Specific State of Failure Stress and the Material Type

It will be helpful to rewrite the failure criteria as a hybrid combination of the tensor form (1) and (2) and the component form (4) and (5). Let stress be non-dimensionalized as

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{\kappa} = \frac{\sigma_{ij}}{C} \quad (6)$$

Then the failure criteria are

$$\begin{aligned} & \alpha \left(\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} \right) \\ & + (1 + \alpha) \left\{ \frac{1}{2} \left[\left(\hat{\sigma}_{11} - \hat{\sigma}_{22} \right)^2 + \left(\hat{\sigma}_{22} - \hat{\sigma}_{33} \right)^2 + \left(\hat{\sigma}_{33} - \hat{\sigma}_{11} \right)^2 \right] \right. \\ & \left. + 3 \left(\hat{\sigma}_{12}^2 + \hat{\sigma}_{23}^2 + \hat{\sigma}_{31}^2 \right) \right\} \leq 1 \end{aligned} \quad (7)$$

and

$$\hat{\sigma}_1 \leq \frac{1}{1 + \alpha} \quad \text{if } \alpha > 1 \quad (8)$$

The advantage of this form is that while the stresses are specified in terms of components, the non-dimensional failure property α , (3), can be used to scan different material types as α varies from 0 to ∞ . In effect, property α will be used to designate the material type. The objective now is to determine whether a given mode of failure is ductile or brittle according to the specification of the state of the material given by α in (3) and the specified homogeneous state of stress of interest.

Begin with the two-dimensional biaxial stress state involving σ_{11} and σ_{22} with all other stress components as vanishing. The criterion (7) becomes

$$\alpha \left(\hat{\sigma}_{11} + \hat{\sigma}_{22} \right) + (1 + \alpha) \left(\hat{\sigma}_{11}^2 - \hat{\sigma}_{11} \hat{\sigma}_{22} + \hat{\sigma}_{22}^2 \right) \leq 1 \quad (9)$$

The fracture cutoff criterion is still given by (8), with σ_1 as the largest principle stress.

The biaxial failure criteria (8) and (9) give the failure forms shown in Fig. 1 for various values of α . These four figures are illustrative and drawn to approximate scale. The fracture cutoff behavior first begins to intercede at $\alpha = 1$, becomes quite prominent at $\alpha = 2$ and becomes very dominant at large values of α . Cast iron is a good material example at $\alpha = 2$ and ceramics are materials with large values of α , Christensen [1]. The case $\alpha = 0$ is that of a very ductile metal controlled by the Mises criterion.

The boundary between ductile and brittle regions of behavior is taken as the dotted lines shown in Fig. 1c and 1d. Thus this boundary is at the edge of the fracture cutoff zones, as shown. The brittle zone is toward the tension-tension quadrant in Fig. 1 while the ductile zone is toward the compression-compression quadrant. The brittle zone is controlled by two separate brittle fracture mechanisms. The fracture cutoff form shown in Fig. 1 is due to the usual crack opening mode of fracture involving a single stress component. The remaining part of the brittle zone in the first quadrant is of a more involved interaction effect between the stress components. In Fig. 1b, the fracture cutoff zones have shrunk to two points. It will be useful to evaluate the position of the fracture cutoff zone in terms of parameter α . This position coordinate is given by coordinate ζ as shown in Fig. 1. The yield/failure function is given by (9) and the fracture form by (8). Take (8) as

$$\hat{\sigma}_{22} = \frac{1}{1 + \alpha} \quad (10)$$

Substitute (10) into (9) to find

$$\hat{\sigma}_{11} = \frac{1 - \alpha}{1 + \alpha} \quad (11)$$

Relations (10) and (11) give the location of the intersection of the yield/failure surface and the fracture cutoff in the second quadrant defining the D-B criterion. The similar point in the fourth quadrant is given by

$$\left. \begin{aligned} \hat{\sigma}_{11} &= \frac{1}{1+\alpha} \\ \hat{\sigma}_{22} &= \frac{1-\alpha}{1+\alpha} \end{aligned} \right\} \quad (12)$$

From (10)-(12) the position coordinate ζ is found to be

$$\zeta = \frac{2 - \alpha}{\sqrt{2}(1 + \alpha)} \quad (13)$$

The line defined by ζ , (13), shown in Fig. 1 separates the ductile and brittle regions. Assuming that this expression for ζ in the range $\alpha > 1$ also smoothly continues in the range $\alpha < 1$ then this gives the tangent condition at $\alpha = 0$ shown in Fig. 1a.

Next, define any two points on the failure/yield surface (9) by another line parallel to the line given by ζ in Fig. 1. Designate its position coordinate by η . From the basic geometry it follows that

$$\eta = \frac{1}{\sqrt{2}} \left(\hat{\sigma}_{11} + \hat{\sigma}_{22} \right) \quad (14)$$

where $\hat{\sigma}_{11}$ and $\hat{\sigma}_{22}$ satisfy (9). This line designated by coordinate η , parallel to that designated by ζ in Fig. 1 must be to the left of the dotted lines in Fig. 1 for ductile behavior to exist. This condition is then stated as

$$\eta < \zeta \quad , \quad \text{Ductile} \quad (15)$$

Combining (13)-(15) gives the 2-D criterion as

$$\hat{\sigma}_{11} + \hat{\sigma}_{22} < \frac{2 - \alpha}{1 + \alpha} \quad , \quad \text{Ductile, 2-D} \quad (16)$$

where $\hat{\sigma}_{11}$ and $\hat{\sigma}_{22}$ must satisfy (9) for the specified stress state of interest. This then is the ductile–brittle criterion for a particular stress state under 2-D conditions. Before discussing this criterion it is best to obtain the corresponding generalization of it.

Now consider three dimensional stress states. The previous 2-D case can be viewed as states in 3-D principle stress space $\hat{\sigma}_{11}$, $\hat{\sigma}_{22}$ and $\hat{\sigma}_{33}$ with one of the principal stresses as vanishing. In the 2-D case the D-B criterion (16) is seen to have the stress state specified in terms of its mean normal stress, or the first stress invariant. This feature is also evident in Fig. 1. For a particular state of the material the boundary between ductile and brittle behavior occurs at a particular value of the mean normal stress for the applied stress state. This same characteristic will be generalized to any 3-D stress state. As discussed in the Introduction, the failure surface (7) in principal stress space is a paraboloid of revolution intersected by three planes normal to the principle stress axes, representing the fracture cutoff effect (8). The mean normal stress (first stress invariant) defines a plane normal to the line that makes equal angles to axes σ_1 , σ_2 , and σ_3 . This plane intersects the failure surface and separates it into two regions, one of ductile failure and one of brittle failure. The location of this 3-D boundary plane is taken such that it intersects the coordinate planes σ_1 , σ_2 , and σ_2 , σ_3 and σ_3 , σ_1 according to the method already established in the 2-D stress state case. This unambiguously defines the location of the boundary between the ductile and brittle regions of the 3-D failure surface specified in terms of the mean normal stress.

Carrying out this 3-D procedure, the criterion for the determination of the failure mode type is found to be given by the direct generalization of (16) as

$$\begin{aligned} \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} < \frac{2 - \alpha}{1 + \alpha} \quad , \quad \text{Ductile} \\ \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} > \frac{2 - \alpha}{1 + \alpha} \quad , \quad \text{Brittle} \end{aligned} \quad (17)$$

where the stress state of interest involved in the left hand side of (17) must satisfy the failure criterion (7). The first stress invariant (three times the mean normal stress) on the left side of (17) is specified by the applied stress state and the right side of (17) designates the material type, through α . This then is the ductile-brittle criterion as determined by the stress state and the α value type of the material according to the present method. The criterion in (17) is invariant to the dimensionality, being of the same form for 1-D, 2-D and 3-D cases.

In addition to the D-B criterion (17) related to the yield-failure criterion (7), it also must be remembered that the fracture cutoff regions themselves are of inherently brittle behavior. The overall brittle region then consists of the front part of the paraboloid failure surface specified by (17) along with the three fracture cutoff surfaces that penetrate the ductile region somewhat further, giving a type of three lobed surface in that region.

The criterion (17) for the failure mode type involves the mean normal stress part of the applied stress state. If in (7) there is no mean normal stress effect, then that would require $\alpha = 0$ and the resulting behavior is completely ductile, as governed by the Mises criterion. There must be a mean normal stress effect in this failure criterion for brittle behavior to ensue. Now some examples will be given to illustrate the method and reveal specific results.

The uniaxial stress case with $\hat{\sigma}_{11} \neq 0$ and all other components as vanishing leaves (17) as

$$\hat{\sigma}_{11} < \frac{2 - \alpha}{1 + \alpha}, \quad \text{Ductile} \quad (18)$$

For tensile stress, the solution from (9) is

$$\hat{\sigma}_{11}^T = \frac{1}{1 + \alpha} \quad (19)$$

Combining (19) into (17) then gives

$$\alpha < 1, \quad \text{Ductile} \quad (20)$$

Thus uniaxial tension is of ductile failure if $\alpha < 1$ and it is brittle if $\alpha > 1$.

For uniaxial compressive stress, the solution from (7) is

$$\hat{\sigma}_{11}^C = -1 \quad (21)$$

which combined with (18) gives

$$-1 < \frac{2 - \alpha}{1 + \alpha}, \quad \text{Ductile} \quad (22)$$

which is always satisfied except as $\alpha \rightarrow \infty$. Thus uniaxial compression is always of ductile failure according to the present treatment.

A state of shear stress is given by

$$\hat{\sigma}_{11} = -\hat{\sigma}_{22} \quad (23)$$

Inserting (23) into criterion (17) gives

$$\alpha < 2 \quad , \quad \text{Ductile} \quad (24)$$

Thus shear stress produces ductile failure if the state of the material is $\alpha < 2$ but it produces brittle failure if $\alpha > 2$.

Further cases readily show how a superimposed pressure can convert what is normally thought of as a brittle material into a ductile material for a specific stress state. Although the results obtained for simple stress states may be fairly obvious, more complex stress states have results from (17) that are not obvious.

Now several specific cases will be summarized, based upon the criterion (17) and as illustrated in the previous examples. The D-B criteria will be specified in terms of T/C values since uniaxial tensile and compressive failure values in this ratio are readily recognizable for different materials.

Uniaxial Tension

$$\frac{1}{2} < \frac{T}{C} \leq 1 \quad : \quad \text{Ductile}$$

$$0 < \frac{T}{C} < \frac{1}{2} \quad : \quad \text{Brittle}$$

Uniaxial Compression

$$\text{All } \frac{T}{C} \quad : \quad \text{Ductile}$$

$$\frac{T}{C} \rightarrow 0 \quad : \quad \text{Borderline}$$

Eqi-Biaxial Tension

$$0 < \frac{T}{C} < 1 \quad : \quad \text{Brittle}$$

$$\frac{T}{C} = 1 \quad : \quad \text{Borderline}$$

Eqi-Biaxial Compression

$$\text{All } \frac{T}{C} \quad : \text{ Ductile}$$

Eqi-Triaxial Tension

$$\text{All } \frac{T}{C} \quad : \text{ Brittle}$$

Eqi-Triaxial Compression

No Yielding or Failure

Simple Shear

$$\frac{1}{3} < \frac{T}{C} \leq 1 \quad : \text{ Ductile}$$

$$0 < \frac{T}{C} < \frac{1}{3} \quad : \text{ Brittle}$$

For a given material having a particular value of α this methodology can be used to determine the expected type of mode of failure for any given stress state. This is of considerable interest since the usual sudden nature of brittle type failures generally have more severe consequences than those ductile type failures. Many specific materials examples for the uniaxial tension case were examined by Christensen [4]. The uniaxial tension case given above corresponds to what in Ref. [1] was described as the D-B criterion.

It is interesting to consider the case of the failure mode type shown above for a simple shear state of stress, this being $\alpha < 2$ for ductile failure. In the case of uniaxial tension $\alpha < 1$ gives ductile failure. Comparison of these two cases then shows that the class of ductile materials in shear is larger than the class of ductile materials in uniaxial tension.

The case of uniaxial compression is especially interesting. The results here show this failure mode to be of ductile type for all materials in this study class. However, in the extreme limit $T/C \rightarrow 0$, representing ultimately damaged materials, then the failure type is borderline between ductile and brittle. There have been research studies which characterize uniaxial compressive failure as being some variation of brittle fracture. The present approach gives the perhaps controversial result that brittle fracture in uniaxial compression is not possible, except in the limiting case. However, examples with the present methodology also show that materials that are normally thought of as being brittle, such as ceramics (with α about 10) under a stress state of major unidirectional

compression, but with a slight transverse tensile stress do cross over to brittle failure. Thus predominately but not totally compressive stress states can cause brittle failure. It is necessary to examine each particular stress state to ascertain its type of failure; intuition can be a misleading guide. An important example of a non-intuitive result is that for a tensile equi-biaxial stress state, which is always of brittle failure type even though a particular material may be thought of as being nominally ductile. Thin shell pressure vessels provide an example where such situation can arise and likely warrant special consideration because of the implied fracture danger.

Failure Surface Orientation

The orientation of a failure surface for a material loaded to failure is intimately related to the D-B state of the material. Materials that are normally considered to be brittle typically fail at 90° relative to the load direction for uniaxial tension. A failure angle analysis will now be developed with the intention to examine its relationship to ductile-brittle behavior generally, and specifically with respect to results in the preceding section.

The onset of failure in the present context represents the termination of the reversible elastic state of deformation and the onset of irreversible deformation. The associated flow rule of plasticity is taken to describe the irreversible components of the deformation, as

$$\dot{\varepsilon}_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}} \quad (25)$$

where $\dot{\varepsilon}_{ij}^p$ is the increment of inviscid plastic strain, and $f()$ is the yield/failure function shown in (7) with λ as a proportionality factor. It is well established that the flow form (25) does not apply for granular materials, such as soils. It is, however, taken to apply here for application to materials which have strong degree of physical coherence and structural integrity established through atomic bonds. This includes materials ranging in type from ideally ductile metals through brittle ceramics.

The orientation of the failure surfaces will now be found for states of uniaxial tension and compression. Using the yield/failure function (7), with nondimensional stress in the flow rule (25) with $\hat{\sigma}_{11}$ being the uniaxial stress then gives for uniaxial tension

$$\begin{aligned}\dot{\varepsilon}_{11}^p &= \lambda(2 + \alpha) \\ \dot{\varepsilon}_{22}^p &= \dot{\varepsilon}_{33}^p = \lambda(-1 + \alpha)\end{aligned}\tag{26}$$

and for uniaxial compression

$$\begin{aligned}\dot{\varepsilon}_{11}^p &= -\lambda(2 + \alpha) \\ \dot{\varepsilon}_{22}^p &= \dot{\varepsilon}_{33}^p = \lambda(1 + 2\alpha)\end{aligned}\tag{27}$$

where λ is now nondimensional.

Axis 1 is in the direction of the applied stress, with axes 2 and 3 normal to it. Now take a rotated coordinate system 1', 2', and 3' such that axis 1' lies in the plane of the failure surface, whatever orientation that may be. Take this as a rotation about axis 3 such that axis 2' is now normal to the failure surface. Let angle β be the angle of the coordinate rotation thus β is the angle between the applied stress direction and the failure surface. Using the tensor transformations, then in the tensile case relations (26) give the rotated components as

$$\begin{aligned}\frac{\left(\dot{\varepsilon}_{11}^p\right)'}{\lambda} &= (2 + \alpha)\cos^2\beta - (1 - \alpha)\sin^2\beta \\ \frac{\left(\dot{\varepsilon}_{22}^p\right)'}{\lambda} &= (2 + \alpha)\sin^2\beta - (1 - \alpha)\cos^2\beta \\ \frac{\left(\dot{\varepsilon}_{12}^p\right)'}{\lambda} &= 3\sin\beta\cos\beta\end{aligned}\tag{28}$$

In order to determine the failure angle, β , take the increment of plastic strain in the plane of the failure surface as non-active and vanishing, while the other two increments in (28) are allowed to grow and flow in an unlimited manner implying failure. Thus take

$$\begin{pmatrix} \bullet p \\ \varepsilon_{11} \end{pmatrix}' = 0 \quad (29)$$

which with (28) gives

$$\tan \beta = \sqrt{\frac{2 + \alpha}{1 - \alpha}}, \quad \alpha \leq 1 \quad (30)$$

It is seen that (30) gives $\beta = 90^\circ$ at $\alpha = 1$, reaching the limit. For values of $\alpha > 1$, the fracture cutoff effect (8) intercedes and brittle fracture with the same procedure using (25) gives

$$\beta = 90^\circ, \quad \alpha \geq 1 \quad (31)$$

In the uniaxial compressive case, the same tensor transformation procedure, but using (27), gives the plasticity components as

$$\begin{aligned} \frac{\begin{pmatrix} \bullet p \\ \varepsilon_{11} \end{pmatrix}'}{\lambda} &= -(2 + \alpha)\cos^2 \beta + (1 - 2\alpha)\sin^2 \beta \\ \frac{\begin{pmatrix} \bullet p \\ \varepsilon_{22} \end{pmatrix}'}{\lambda} &= -(2 + \alpha)\sin^2 \beta + (1 - \alpha)\cos^2 \beta \\ \frac{\begin{pmatrix} \bullet p \\ \varepsilon_{12} \end{pmatrix}'}{\lambda} &= -3(1 + \alpha)\sin \beta \cos \beta \end{aligned} \quad (32)$$

Again, setting the increment of irreversible strain $\begin{pmatrix} \bullet p \\ \varepsilon_{11} \end{pmatrix}'$ equal to zero such that this

component in the failure plane cannot grow in an unlimited manner in the way that the other two out of plane components grow in failure, gives

$$\tan \beta = \sqrt{\frac{2 + \alpha}{1 + 2\alpha}} \quad (33)$$

The results of this failure angle analysis are shown in Tables 1 and 2. The results correctly predict that in the ductile limit, $\alpha = 0$, the failure angle is the octahedral angle

$\beta = \tan^{-1} \sqrt{2} = 54.7^\circ$, Nadai [10]. The $\alpha \rightarrow \infty$ limits are given by 90° in the case of tension and $\beta = \tan^{-1} (1 / \sqrt{2}) = 35.3^\circ$ in the case of compression.

It is interesting to compare the failure angle behavior just found with that predicted by the Coulomb-Mohr method. From Paul [6] the Coulomb-Mohr failure angles are given by

$$\beta = 45^\circ + \frac{\phi}{2}, \text{ Tension}$$

and

$$\beta = 45^\circ - \frac{\phi}{2}, \text{ Compression}$$

(34)

where

$$\phi = \sin^{-1} \left(\frac{\frac{C}{T} - 1}{\frac{C}{T} + 1} \right) \quad (35)$$

In the ductile limit $C=T$, the Coulomb-Mohr result gives $\beta = 45^\circ$ and does not correctly predict the octahedral angle.

The present failure angle predictions and those from Coulomb-Mohr are shown in Fig. 2. It is seen that they produce fundamentally different failure surface orientations. Only in the limit, $\alpha \rightarrow \infty$ does the Coulomb-Mohr method correctly predict a fracture angle of $\beta = 90^\circ$ in tension. Typical glassy, brittle polymers such as polystyrene, PMMA, and some epoxies can have uniaxial tension to compression ratios as low as about 1/2. Experimental results show these materials to fail brittly in tension with a failure angle of $\beta = 90^\circ$, Berry [11], see also Ward [12]. Coulomb-Mohr predicts this failure angle from (34) as 54.7° . The present method (30), (31) correctly predicts $\beta = 90^\circ$. The Coulomb-Mohr method also predicts $\beta \rightarrow 0$ as $\alpha \rightarrow \infty$ in compression whereas the present method predicts the limiting angle as $\beta = 35.3^\circ$. Most importantly, the Coulomb-Mohr method completely misses a failure angle manifestation of the ductile-brittle transition at $\alpha = 1$, whereas the present method shows a strong change in failure mode type in tension at $\alpha = 1$, Fig. 2. In uniaxial compression, Fig. 2, the present failure angle

behavior shows no transition feature and thereby is consistent with the previously found result that this mode of failure is always ductile.

Some Conclusions

The analysis given here predicts the failure state of uniaxial compression as always being ductile, whereas there is a considerable body of opinion that it is, or can be brittle. A close look at the results shows that in the limit as α goes to infinity this failure type becomes borderline between ductile and brittle. This condition is also shown in Fig. 1d where the edge of the brittle region is so close to the point of uniaxial compressive failure that it probably has a dominating effect over it, causing uniaxial compression to effectively be of the brittle type in the range of large α . Thus, the present analysis suggests that brittle materials such as possibly ceramics and certainly many minerals possess a brittle failure type in uniaxial compression even though the compressive failure for different materials satisfying the Mises criterion is of ductile type.

There is considerable evidence for the existence of the D-B transition in uniaxial tension as being at the value $\alpha = 1$. Such data sources were collected in Christensen [4]. Most other stress states are lacking in applicable data. It would be especially relevant to investigate the associated D-B regions for simple shear.

Geomechanics work sometimes utilizes combinations of a Rankine (principal stress) criterion with the Coulomb-Mohr criterion. Further work could possibly compare that approach with the present approach as regards brittle versus ductile behavior, although it is not clear how to extract a D-B criterion from the geomechanics approach.

Although every different stress state has its own distinctive ductile-brittle characterization, one particular stress state stands out as overwhelmingly more important than the others. This is the case of uniaxial tension. In comparing different materials, it is uniaxial tension that gives the most decisive and physically meaningful measure of distinction in a ductile versus brittle sense. The D-B transition for uniaxial tension being at $\alpha = 1$ ($T/C = 1/2$) makes it extremely easy to determine and to interpret where a given material stands in this regard.

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Tables

Table 1 Failure Angles, Uniaxial Tension, Eqs.(30) and (31)

Table 2 Failure Angles, Uniaxial Compression, Eq. (33)

Figures

Figure 1 Biaxial Stress State Failure Forms

Figure 2 Failure Angle Predictions

Material	α	$\frac{T}{C}$	Tensile β Degrees
Ductile Metals	0	1	54.7
Ductile Metals & Polymers	$\frac{1}{4}$	$\frac{4}{5}$	60
Ductile Polymers	$\frac{1}{2}$	$\frac{2}{3}$	65.9
Transition, Glassy Polymers	1	$\frac{1}{2}$	90
Brittle Materials	>1	$<\frac{1}{2}$	90

Table 1. Failure Angles, Uniaxial Tension, Eqs. (30) and (31)

Material	α	$\frac{T}{C}$	Compressive β Degrees
Ductile Metals	0	1	54.7
Ductile Polymers	$\frac{1}{2}$	$\frac{2}{3}$	48.2
Transition, Glassy Polymers	1	$\frac{1}{2}$	45
Cast Iron	2	$\frac{1}{3}$	41.8
Ceramics	10	$\frac{1}{11}$	37.1
Brittle Limit	$\rightarrow\infty$	$\rightarrow 0$	35.3

Table 2. Failure Angles, Uniaxial Compression, Eq. (33)

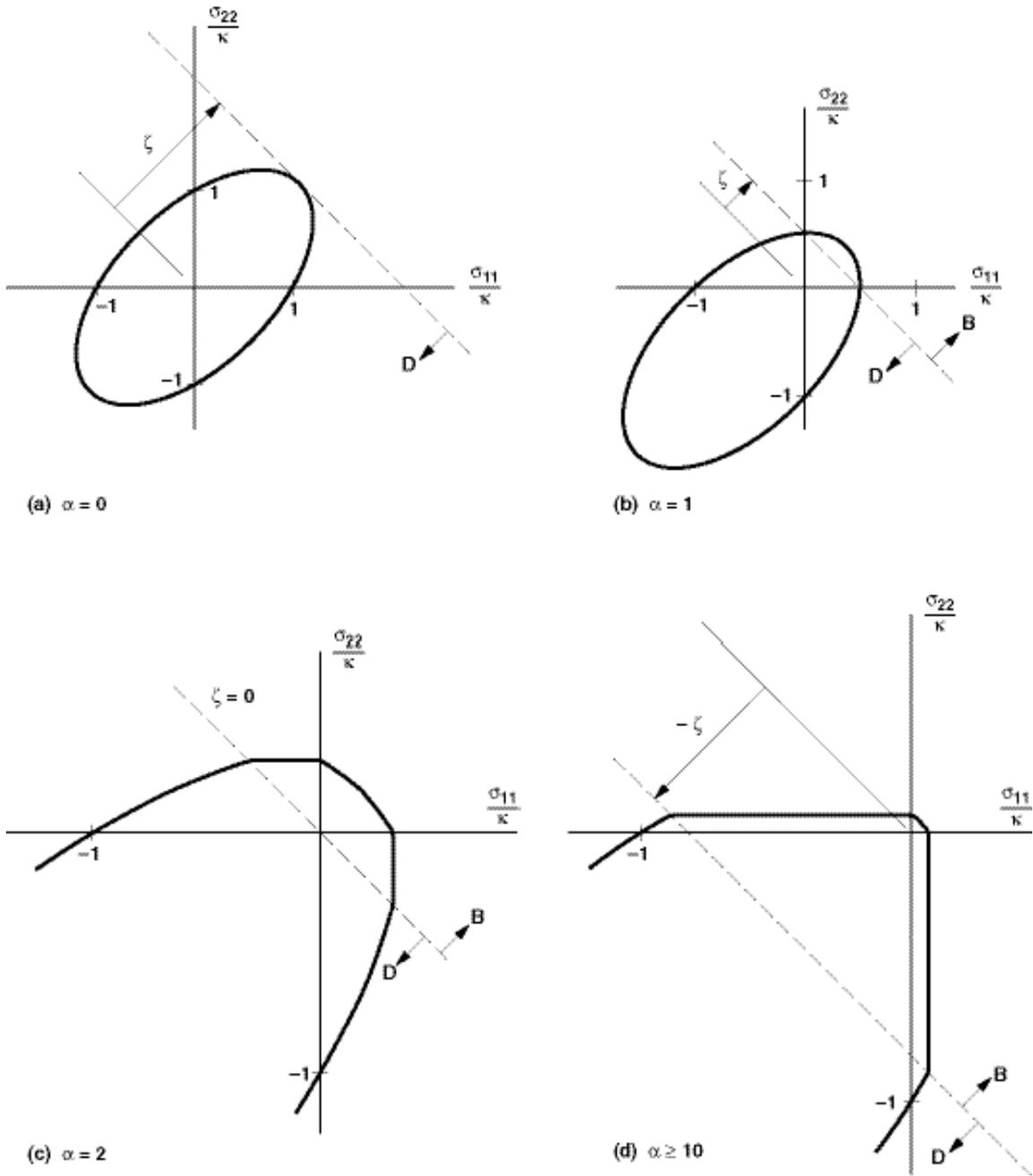


Figure 1. Biaxial Stress State Failure Forms

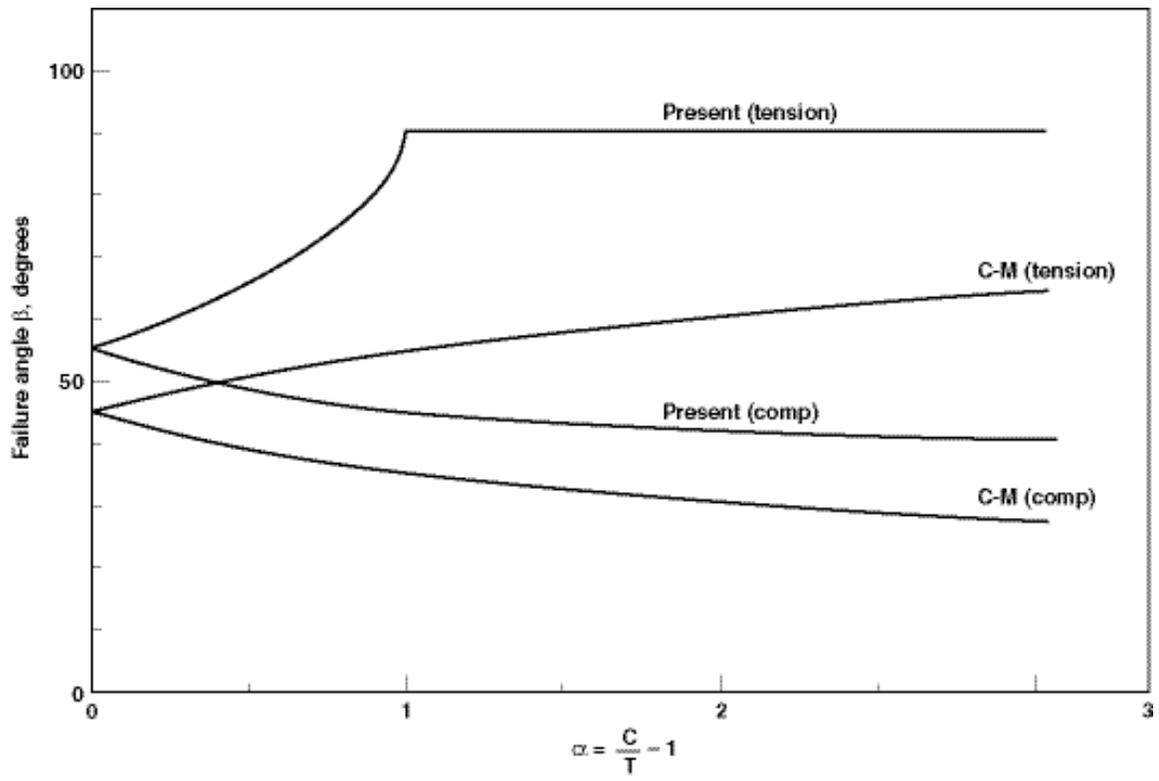


Figure 2. Failure Angle Predictions