XIV. Completion And Closure On Failure Criteria For Unidirectional Fiber Composites

Background

In Section III and in Christensen [1] the polynomial invariants method was used to develop failure criteria for aligned fiber composite materials. For the highly anisotropic cases such as with carbon fibers in a polymer matrix phase, the failure theory naturally partitioned into two separate failure criteria, one being matrix controlled and one being fiber controlled. There resulted seven individual failure properties needed to calibrate the theory.

One of the seven calibrating failure properties was eliminated on an order of magnitude basis, leaving four properties as being matrix controlled and two as being fiber controlled. The four matrix controlled failure properties are T_{22}, C_{22}, S_{23}, and S_{12}. The first three are the transverse failure properties, and therein was found to lie an unusual problem. The failure criterion revealed an apparent extreme sensitivity to the size of the transverse shear failure property S_{23} relative to the sizes of the two transverse uniaxial failure properties T_{22} and C_{22}.

A micromechanics analysis was conducted in Section IX to determine S_{23} in terms of T_{22} and C_{22} in order to overcome the sensitivity problem. But this then brings in a subsequent question as to whether it is a legitimate operation to eliminate one of the matrix controlled failure properties in this manner, or is it just an artificial device to get around a perceived but not real sensitivity problem.

All of these matters are opened up here, closely examined, and ultimately resolved with finality. The failure properties sensitivity problem will turn out to provide the missing insight needed to complete the development of the failure criterion.

Derivation

The controlling failure criteria for unidirectional fiber composite materials, [1], are given by the matrix controlled form

$$
\begin{aligned}
&\left(\frac{1}{T_{22}}-\frac{1}{C_{22}}\right)(\sigma_{22}+\sigma_{33})+ \\
&\frac{1}{T_{22}C_{22}}(\sigma_{22}+\sigma_{33})^2+\frac{1}{S_{23}^2}(\sigma_{23}^2-\sigma_{22}\sigma_{33})+\frac{1}{S_{12}^2}(\sigma_{12}^2+\sigma_{31}^2)\le 1
\end{aligned}
\tag{1}
$$

and the fiber controlled form

$$
\left(\frac{1}{T_{11}}-\frac{1}{C_{11}}\right)\sigma_{11}+\frac{1}{T_{11}C_{11}}\sigma_{11}^2\le 1 \tag{2}
$$

or its equivalent common form

$$
-C_{11}\le\sigma_{11}\le T_{11}
$$

In the matrix controlled criterion (1) it is necessary to have

$$
S_{23}^2\ge\frac{1}{4}T_{22}C_{22} \tag{3}
$$

in order to always have real roots from the quadratic form of the failure criterion. The fiber controlled criterion (2) is of the well known normal stress form, but it has a rational basis in the derivation rather than just the usual empirical appeal.

A very interesting problem quickly arises from these failure criteria forms. Often the reported failure data for T_{22}, C_{22}, and S_{23} are found to violate the restriction (3). Does this mean that the matrix controlled failure criterion (1) is ill posed and thereby unacceptable? The failure properties, especially C_{22} and S_{23}, are notoriously difficult to determine with reasonably high accuracy. This then enlarges the problem to one or other of the two propositions: (i) either the failure criterion (1) is improper or (ii) the materials failure properties must be determined to an extremely, perhaps unattainably, high level of accuracy. This work explores this difficulty and seeks an acceptable direction for moving forward.

The problem just outlined does not involve the axial shear strength property S_{12} so the reduced relevant failure criterion (1) under examination is given by

$$\left(\frac{1}{T_{22}}-\frac{1}{C_{22}}\right)(\sigma_{22}+\sigma_{33})+\frac{1}{T_{22}C_{22}}(\sigma_{22}+\sigma_{33})^2+\frac{1}{S_{23}^2}\left(\sigma_{23}^2-\sigma_{22}\sigma_{33}\right)\leq 1 \tag{4}$$

The issue of the behavior of failure criterion (4) comes into the sharpest focus when the stresses are taken to be those of equal biaxial stress with

$$\begin{aligned}&\sigma_{22}=\sigma_{33}=-P_{22}\\&\sigma_{23}=0\end{aligned} \tag{5}$$

and where positive P_{22} is the compressive failure level. Thus (4) becomes

$$\left(4-\frac{T_{22}C_{22}}{S_{23}^2}\right)\frac{P_{22}^2}{T_{22}C_{22}}-2\left(\frac{1}{T_{22}}-\frac{1}{C_{22}}\right)P_{22}-1=0 \tag{6}$$

In this form it is seen how the restriction (3) must come into consideration.

Let the failure strengths P_{22} and S_{23} be nondimensionalized as follows

$$\begin{aligned}&\hat{P}_{22}=\frac{P_{22}}{C_{22}}\\&\hat{S}_{23}^2=\frac{S_{23}^2}{T_{22}C_{22}}\end{aligned} \tag{7}$$

Then (6) becomes

$$\left(4-\frac{1}{\hat{S}_{23}^2}\right)\hat{P}_{22}^2-2\left(1-\frac{T_{22}}{C_{22}}\right)\hat{P}_{22}-\frac{T_{22}}{C_{22}}=0 \tag{8}$$

Alternatively $\hat{S}_{23}^2$ can be determined directly from (8) to give

$$\hat{S}_{23}^2=\frac{\hat{P}_{22}^2}{4\hat{P}_{22}^2-2\left(1-\frac{T_{22}}{C_{22}}\right)\hat{P}_{22}-\frac{T_{22}}{C_{22}}} \tag{9}$$

Although the quadratic form (8) gives two roots the immediate concern here is with the case of the positive root, the compressive eqi-biaxial stresses at failure.

From (8) the functional dependence of $\hat{P}_{22}$ versus $\hat{S}_{23}^2$ can be found for specified values of T_{22}/C_{22}. The limiting envelopes at $T_{22}/C_{22}=0$ and 1 are as shown in Fig. 1.

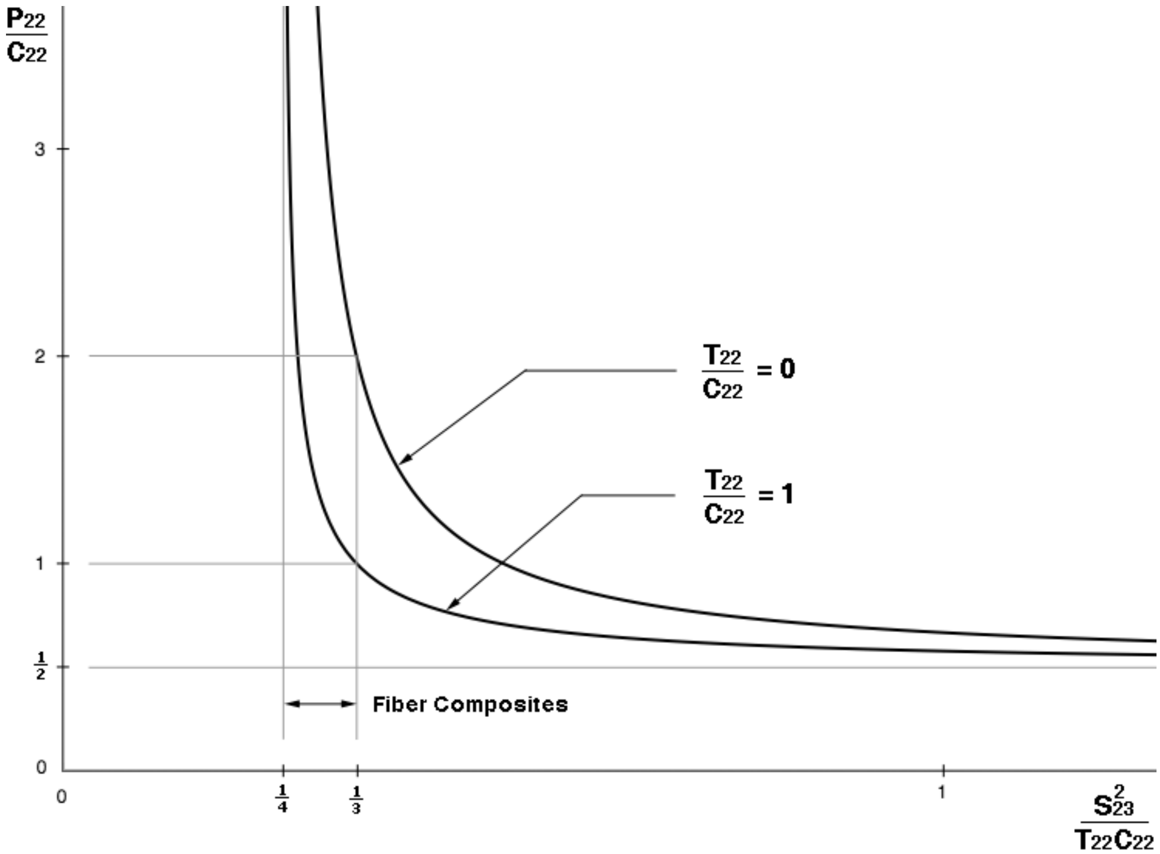

Fig. 1 Failure envelopes from (8)

The two asymptotes at $\hat{P}_{22} = 1/2$ and $\hat{S}_{23}^2 = 1/4$ shown in Fig. 1 are independent of the value of T_{22}/C_{22}.

The key to proceeding further is to determine what region of the failure map shown in Fig. 1 is occupied by highly anisotropic, full density fiber composite materials. As it stands the entire failure map is for all transversely isotropic materials that are also highly anisotropic. Very stiff and very strong fiber composites are in the left most region of Fig. 1. Materials to the right could be highly porous transversely isotropic materials such as ones that have a unidirectional packing of bonded “soda straw” type of forms.

Carbon fiber composites are to the left of

$$S_{23}^2 = \frac{1}{3} T_{22} C_{22}$$

in Fig. 1 because such composites are generally expected to have $P_{22} > 2C_{22}$ and certainly must have $P_{22} \geq C_{22}$, placing them in the upper region on the left hand side of Fig. 1, as shown.

From Fig. 1 it is now graphically seen why it is necessary that restriction (3) be satisfied. Furthermore it is immediately obvious that in the fiber composites region which is given by

$$\frac{1}{4} T_{22} C_{22} \leq S_{23}^2 \leq \frac{1}{3} T_{22} C_{22} \qquad (10)$$

very small changes in the shear strength property S_{23} produce very large changes in the compressive eqi-biaxial failure property P_{22}. This characteristic places severe and probably impossible demands upon the experimental accuracy required in determining S_{23} for fiber composites. Not only do reported data often violate the lower limit in (10), as already mentioned, other reported data often violate the upper limit in (10).

Consider briefly the reverse situation, that is if the interest were in very different types of materials modeled by failure in the right hand side of Fig. 1. Then the failure behavior would be of the converse type. There would still be great sensitivity of the failure properties, but very small changes in P_{22} would give extremely large changes in S_{23}. This shows the importance of carefully determining the limits of behavior for any particular class of materials. High strength and high stiffness fiber composites are confined to the left most portion of Fig. 1

The original question posed at the beginning is now answered, there is no inherent problem with the legitimacy of the failure criterion (1) or its reduced form (4). However, this then creates a new uncertainty. How should the property S_{23} be determined when it cannot be determined directly with the required accuracy. In principle one could determine P_{22} and then using (9) S_{23} would be found. That would certainly overcome the sensitivity problem. Regrettably that is not a practical solution because it is a difficult experiment to directly determine P_{22} and it could never be done on a routine basis. Fortunately this complex situation is not completely blocked, there remains an alternative approach.

The sensitivity problem with S_{23} was recognized earlier, and a micromechanics analysis was used in Section IX to determine S_{23} theoretically. Specifically, the isotropic material failure theory was used to determine the failure of the matrix phase at the micro-scale that then lead to the macroscopic result

$$S_{23}^2 = \frac{2}{7} T_{22} C_{22} \quad at \quad \frac{T_{22}}{C_{22}} = \frac{1}{3} \tag{11}$$

It was a long and complicated derivation to find S_{23} and it was only determined for the most common value of the transverse strengths ratio $T_{22}/C_{22} = 1/3$. So the micromechanics result (11) represents an estimate for S_{23} with respect to the entire range of possible T_{22}/C_{22} values, even though it is an exact result at one particular value. Perhaps it is a very good estimate, but it's still only an estimate. The task now is to determine how useful or not useful the single value (11) actually is.

The approach to be followed here is to take the general form

$$S_{23}^2 = \lambda T_{22} C_{22} \tag{12}$$

where

$$\lambda = \lambda\left(\frac{T_{22}}{C_{22}}\right)$$

and determine the function $\lambda\left(T_{22} / C_{22}\right)$. Then the value of (11) can be assessed and/or replaced by a new and more general result for (12).

The known information to be used in determining λ is its value at the one specific point

$$\lambda = \frac{2}{7} \quad at \quad \frac{T_{22}}{C_{22}} = \frac{1}{3} \tag{13}$$

and its range given by (10), thus

$$\frac{1}{4} \le \lambda \le \frac{1}{3} \tag{14}$$

The limits on T_{22} /C_{22} are

$$0 \le \frac{T_{22}}{C_{22}} \le 1 \tag{15}$$

It is easily reasoned that the limits shown in (14) and (15) associate with each other through

$$\begin{aligned} &\frac{T_{22}}{C_{22}} = 0 \quad associates\ with \quad \lambda = \frac{1}{3} \\ &\frac{T_{22}}{C_{22}} = 1 \quad associates\ with \quad \lambda = \frac{1}{4} \end{aligned} \tag{16}$$

If they were taken the alternative way it would lead to physically irrational and unacceptable results.

Take the general form for λ as given by

$$\lambda = \frac{1+\alpha \dfrac{T_{22}}{C_{22}}}{\beta + \gamma \dfrac{T_{22}}{C_{22}}} \tag{17}$$

This is the usual form found for the effective stiffness properties in many different types of composite materials. It is now utilized here for strengths.

The three parameters in (17) are to be found from the three conditions stated by (13)-(16), giving α=1, β=3, and γ=5. With λ so determined, the final general form for S_{23} is given by

$$S_{23}^2 = \left(\frac{1+\dfrac{T_{22}}{C_{22}}}{3+5\dfrac{T_{22}}{C_{22}}} \right) T_{22} C_{22} \tag{18}$$

With the explicit result (18) expressing the transverse shear strength in terms of the transverse uniaxial tensile and compressive strengths, then the solution for P_{22} can be found by substituting (18) into (8). And after consolidating many terms there finally results

$$\frac{P_{22}}{C_{22}} = \left(1+\frac{T_{22}}{C_{22}}\right) \pm \sqrt{\frac{\left(1+\dfrac{T_{22}}{C_{22}}\right)\left[1+\dfrac{T_{22}}{C_{22}} - \left(\dfrac{T_{22}}{C_{22}}\right)^2\right]}{\left(1-\dfrac{T_{22}}{C_{22}}\right)}} \tag{19}$$

It is readily shown from (18) and (19) that S_{23} and P_{22} correctly satisfy all the proper limiting case behaviors at T_{22}/C_{22}=0 and 1. Relations (18) and (19) are remarkably simple and concise results considering the complexity of the problem.

Now for comparison purposes it is useful to obtain the solution for P_{22} using the simplified form (11) for S_{23}. Combining (4) and (11) gives

$$\frac{P_{22}}{C_{22}} = 2\left[1-\frac{T_{22}}{C_{22}} \pm \sqrt{1-\frac{3}{2}\left(\frac{T_{22}}{C_{22}}\right)+\left(\frac{T_{22}}{C_{22}}\right)^2}\,\right] \tag{20}$$

The form (20) and its simple antecedent form (4) do not recover the proper limiting results at $T_{22}/C_{22}=0$ and 1. Nevertheless they may still be quite useful over the usual and rather narrow range of values of T_{22}/C_{22} for fiber composites. These possibilities will be examined next.

Evaluation

Next a comparison will be made between the two forms for S_{23}, namely the form directly from micromechanics (11) and the more general form derived here, (18). The comparison also importantly extends to the respective two forms (19) and (20) for P_{22}, the eqi-biaxial stresses at failure, both tensile and compressive. To effect this comparison it is necessary to assign the values of T_{22}/C_{22}. Of course the most common value of $T_{22}/C_{22}=1/3$ will be given, but also the two common and usual limits shown by typical fiber composite materials are namely

$$0.25 \le \frac{T_{22}}{C_{22}} \le 0.40 \tag{21}$$

All three of these cases and comparisons are shown in Table 1 where Eq. (20) is based upon $\hat{S}_{23}^2=2/7$ and (19) is based upon the more general form for S_{23}, (18).

	$\frac{T_{22}}{C_{22}}=\frac{1}{4}$	$\frac{T_{22}}{C_{22}}=\frac{1}{3}$	$\frac{T_{22}}{C_{22}}=0.4$
$\hat{S}_{23}^{2}=\frac{2}{7}$ (11)	0.286	0.286	0.286
$\hat{S}_{23}^{2}$ (18)	0.294	0.286	0.280
$\hat{P}_{22}$ (20) *Tension*	-0.158	-0.230	-0.297
$\hat{P}_{22}$ (19) *Tension*	-0.157	-0.230	-0.301
$\hat{P}_{22}$ (20) *Compression*	3.16	2.90	2.70
$\hat{P}_{22}$ (19) *Compression*	2.66	2.90	3.10

Table 1 Comparison of results based upon the simplified form for $\hat{S}_{23}^{2}$ and the general form for $\hat{S}_{23}^{2}$

From the first two rows in Table 1 it is seen that the comparisons between the S_{23} values are extremely close, further showing the extreme sensitivity of this property. In the center two rows, the tensile values for P_{22} are also extremely close by the two methods. In fact nearly all stress states would be extremely closely specified by the two forms for S_{23}. The only stress state that brings out the differences between the two cases is that of eqi-biaxial compressive failure. As seen from the last two rows in Table 1, the differences between the values for compressive P_{22} are about 13% at the one usual limit and about 19% at the other usual limit. These differences are about the same as the maximum differences between the Mises and Tresca criteria for isotropic materials. Usually differences of these amounts are acceptable for failure assessments. Based upon all of these results, either form for S_{23}, (11) or (18), is perfectly acceptable for general and normal use.

Now an assessment of this overall matrix controlled failure criterion for fiber composites will be given. From Table 1 for the most common case of $T_{22}/C_{22}=1/3$ it is found that in eqi-biaxial stresses the ratio of the tensile failure level to the compressive failure level is given by

$$\frac{P_{22}^{-}}{P_{22}^{+}}=\frac{1}{12.6}$$

Anything much smaller than this value might incorrectly be seen as "masking" an inclination for the material to give $P_{22}^{+}\rightarrow\infty$, which is utterly impossible. On the other hand, anything about the same size as $T_{22}/C_{22}=1/3$ would also not be possible. Although there is no credible experimental data for this condition, the value shown above is a completely reasonable and rational physical result, and represents a general type of affirmation of the failure criterion.

It is well known that superimposed pressure has a profound effect on failure stresses. The formulas for this effect are easily derived where p is taken to be the superimposed pressure. The effects of the pressure on the transverse uniaxial failure stresses and the transverse shear failure stress are found to be given by

$$\left.\frac{d\sigma_{22}^{\pm}}{dp}\right|_{p=0}=(\pm)\frac{\left(2+3\frac{T_{22}}{C_{22}}\right)\left(1-\frac{T_{22}}{C_{22}}\right)}{\left(1+\frac{T_{22}}{C_{22}}\right)^{2}}$$

and (22)

$$\left.\frac{d\sigma_{23}}{dp}\right|_{p=0}=\sqrt{\frac{1+\frac{T_{22}}{C_{22}}}{3+5\frac{T_{22}}{C_{22}}}}\left(\sqrt{\frac{C_{22}}{T_{22}}}-\sqrt{\frac{T_{22}}{C_{22}}}\right)$$

The stresses at failure are those superimposed upon the pressure state p. These results are for the general form for S_{23}, (18). The corresponding results for the simpler form of S_{23}, (11), are of slightly simpler forms but the values resulting from the two forms of S_{23} are extremely close to each other over the normal range (21). Again, this is consistent with the results in Table 1.

In the most important case of $T_{22}/C_{22}=1/3$ then (22) gives

$$\left.\frac{d\sigma_{22}^{\pm}}{dp}\right|_{p=0} = (\pm)\frac{9}{8}$$
$$\left.\frac{d\sigma_{23}}{dp}\right|_{p=0} = \frac{4}{\sqrt{42}} = 0.617 \tag{23}$$

First note that relations (22) show that there is no effect of pressure when $T_{22}/C_{22}=1$, as must be the case. Relations (22) and (23) further show that the pressure reinforcing effect in the uniaxial case is the same for compression as for tension. Even though the tensile strength may be much less than the compressive strength, the reinforcing magnitude depends only upon the increment in pressure. This is in accordance with experience. Lastly, the size of the typical reinforcing effect, as in (23), is large, not negligible. This would seem to be appropriate to this class of behavior which exhibits large P_{22} strengths.

There is no need to independently verify the fiber controlled failure criterion (2). It has been employed for many years on a completely satisfactory basis. There has always been much speculation that the fiber direction failure stress should be coupled with some of the other stress components. But the derivation of (2) showed that this is not the case. The original, intuitive preference for the simple maximum stress form is also the rigorous form that emerges from the polynomial invariants method for high stiffness and high strength fiber composites.

Conclusions

The matrix controlled failure criterion sensitivity to the relative sizes of S_{23}, T_{22}, and C_{22} is a very real, physically inherent, and totally determinative characteristic for fiber composite materials. The interrelationship between these properties is brought into greatest clarity through Fig. 1. The physically permissible range of variation for S_{23}^2 extends only from $(T_{22}C_{22})/4$ to $(T_{22}C_{22})/3$. It would be virtually impossible to experimentally determine S_{23} to the required accuracy within these narrow limits. It is necessary to use theoretical mechanics to find the functional relationship between S_{23} and T_{22} and C_{22}, as has been done here.

Perhaps for some it is surprising, even distressing, that the transverse shear strength is determined by the transverse tensile and compressive strengths. But with further consideration, it would be surprising if it were not so determined. For isotropic materials the shear strength is determined by the tensile and compressive strengths. For unidirectional fiber composites the three transverse strengths are those for a state of transverse isotropy, which is a completely comparable situation, but in two dimensions rather than in three. In both of these isotropic cases physical conditions require the above stated outcomes.

The micromechanics derived form for S_{23} given by (11) at $T_{22}/C_{22}=1/3$, and the more general form for S_{23} given by (18), as derived here for all values of T_{22}/C_{22}, are both shown to be useful for general applications of failure. Either form not only can be but must be used in the matrix controlled failure criterion (1). Either form is consistent with the usual experimental accuracy for properties determination and applications. In the unusual situation where a particular fiber composite materials type is reliably known to be substantially outside the usual range for the T_{22}/C_{22} ratio given by (21), then the more general form (18) should be used.

Either of these forms for S_{23} reduce the total properties count for the failure theory to five, namely T_{11}, C_{11}, T_{22}, C_{22}, and S_{12}. Performing the difficult experiment of determining the transverse shear strength is extraneous and irrelevant. The five failure properties is the same as the number of elastic properties for aligned fiber composites. This balance of failure properties to elastic properties is completely compatible with the circumstance for isotropic materials where there are two elastic properties and two related failure properties, [1].

This five property failure theory for aligned fiber composites is of somewhat unusual status and significance. It could not have been developed if the two property failure theory for isotropic materials had not been derived and verified first. Both theories are among the few having a rational physical basis that inevitably lead to extraordinarily simple forms. Most failure criteria for isotropic materials and for fiber composite materials are constituted by conjectures and data fittings and a considerable to high degree of complexity. Complexity in this context is sometimes mistaken for sophistication.

Finally, it is observed that the matrix controlled failure form (4) for the transverse failure condition reduces to an especially transparent form when the simple S_{23} form (11) is substituted into it. There results in terms of principal stresses,

$$\left(\frac{1}{T_{22}}-\frac{1}{C_{22}}\right)(\sigma_2+\sigma_3)+\frac{1}{T_{22}C_{22}}\left[(\sigma_2-\sigma_3)^2+\frac{1}{2}\sigma_2\sigma_3\right]=1 \qquad (24)$$

It is the last term inside the brackets of (24) that allows and requires finite values for eqi-biaxial compressive failure. Extremely large but not unlimited values for compressive P_{22} and very small values for tensile P_{22} are the primal characteristics for highly anisotropic fiber composite materials. All composites failure theories should be critically examined with respect to these required behaviors.

This completes the development of failure criteria for high stiffness and high strength fiber composite materials. Of broader significance, it also effectively completes the isotropic materials failure theory since the two separate theories coordinate with and reinforce each other. For both isotropic materials and for fiber composite materials the challenges ahead now shift to those of interpretations and applications in the most physically meaningful sense of those terms. For isotropic materials this could mean, among other things, examining the ductile/brittle aspects of failure behavior further because such matters are so overwhelmingly important.

For fiber composite materials the way ahead is less clear. Obviously the ultimate objective is to achieve the application of failure behavior to planar laminated forms and to all types of woven fiber forms. Getting there is the problem. Just as it was herein necessary to fully and completely understand isotropic material failure behavior before attempting the lamina (unidirectional fibers) level, so to it would be necessary to fully assimilate the lamina level before attempting the laminate level. At the present time, the lamina level is still the barrier to further progress.

A new and carefully developed five property failure theory has been given here for the lamina level. There are many other lamina level failure theories in various states of current development. It would seem necessary that all of these lamina level theories be critically compared with each other before attempting to push into laminates. To go into laminates before fully consummating the lamina level understanding would be ill-advised and could lead to yet further confusion.

That has been the continuing problem with fiber composites failure theory all along. It has not proceeded by logical, careful steps of incremental development, each step building upon the previous ones. Rather, everything has been indiscriminately mixed and stirred together, inevitably becoming an almost undecipherable conglomerate. The theoretical failure criteria program is in grave need of some developmental discipline. Otherwise the composites designers and the manufacturing experts will completely give up on the theoretical program and proceed entirely without its and their help.

Reference

1. Christensen, R. M., 2013, The Theory of Materials Failure, Oxford University Press, Oxford, U. K.

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