A Comparative Evaluation of Three Isotropic, Two Property Failure Theories

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Abstract

   Three fundamentally different failure theories for homogeneous and isotropic materials are examined in both the ductile and brittle ranges of behavior. All three theories are calibrated by just two independent failure properties. These three are the Coulomb-Mohr form, the Drucker-Prager form, and a recently derived theory involving a quadratic representation along with a fracture restriction. The three theories are given a detailed comparison and evaluation. The Coulomb-Mohr form and the Drucker-Prager form are found to predict physically unrealistic behavior in some important cases. The present form meets the consistency requirements.

Introduction

The first formulation of a general failure criterion for homogeneous and isotropic materials was given by Coulomb [1] 232 years ago. Looking back, it was a prescient contribution, and it has remained a significant influence every since its conception. It was put into its present form 91 years ago by Mohr [2], and is now known as the Coulomb-Mohr Theory. The next most prominent and actively used or recognized general theory was suggested by Nadai [3] 55 years ago, and perhaps by others earlier, and put into its present form by Drucker and Prager [4] 53 years ago. It is somewhat surprising that these two long-standing and pre-eminent theories have not received a thorough comparison and joint evaluation. Such will be given here, along with a third theory that has been recently developed, but also has some historical lineage, as will be mentioned later.

The theories of materials failure mentioned above are calibrated by two independent parameters or properties. The present work is confined to this two-parameter level because this is the number of failure parameters that may admit interpretation as readily accessible physical properties. Certainly, the Coulomb-Mohr form was intended to admit such interpretations. One can view failure criteria as the means of technically specifying the limits of the elastic region of behavior for materials. Now linear, isotropic elastic materials require the knowledge of two moduli type properties for the characterization of the elastic region. This suggests that two more properties may also suffice to characterize the failure limit of the isotropic, elastic range. These two groups of two properties each offer a non-trivial correlation with the fact that general states of deformation are fully characterized by two states, those of dilatation and distortion. Taking the state of pure dilatation as allowing a positive volume change up to failure, but no failure under hydrostatic compression for homogeneous and isotropic materials, and taking shear as requiring a second failure property for isotropic materials then leads to a two-property failure specification.

For these reasons the present work considers only two property failure forms. All failure forms with three or more parameters or properties have, at least partially, an empirical basis. Any combination of two independent properties tests then calibrates the failure forms considered here for general use. For example, the dilatational and shear

failure properties just discussed could be directly determined by testing. However, by far the most common, most accessible, and most generally useful tests for failure properties are those of uniaxial tension and compression. Uniaxial tension and compression have a certain symmetry as stress state stimuli, while the corresponding magnitudes of the failure responses can have an a-symmetry which reflects fundamental material response mechanisms. These failure properties, specified by "T" and "C", will be used throughout this work.

In the following sections the Coulomb-Mohr and Drucker-Prager theories will be fully compared with each other as well as with a third recently developed theory, Christensen [5]. It is important to effect these comparisons in both the ductile and brittle ranges since any purported general theory must span a broad range of physical effects. In this examining process a variety of stress states will be considered, especially looking for unusual and even physically unacceptable aspects of behavior. It is necessary to begin with clear, concise mathematical statements of the three failure theories.

Coulomb-Mohr Failure Theory

The procedure for the Coulomb-Mohr theory (C-M) uses the Mohr's circle construction as shown in Fig. 1. On the normal stress, σ, versus shear stress, τ, axes plot the circles for uniaxial tension and compression to failure as shown in Fig. 1. Then two linear failure envelopes are taken as just being tangent to the two circles. In order to evaluate the safety for any given three dimensional state of stress, the maximum and minimum principal stresses are used to form a circle on the σ-τ plot. If this circle is inside the failure envelopes then there is no failure, otherwise there is failure of the material. For example, take a state of simple shear stress. Form a circle with its center at the origin in Fig. 1 and just tangent to the failure envelopes. This circle intersects the τ axis at the values of the shear stress at failure, S, given by

$$S = \frac{TC}{T + C} \tag{1}$$

where T and –C are the failure levels in uniaxial tension and compression. The general characteristics of the construction in Fig. 1 are given by

$$x = \frac{TC}{C - T}$$

$$y = \frac{1}{2}\sqrt{CT} \qquad (2)$$

and

$$\sin\phi = \frac{C - T}{C + T}$$

where $T \leq C$.

The procedure just outlined for failure assessment from Fig. 1 can be expressed in analytical form. Let σ_1, σ_2, and σ_3 be the principal stresses with σ_1 being the largest and σ_3 being the smallest in an algebraic sense. From relations (1) and (2) and Fig. 1, the failure criterion can be shown to be given by

$$\frac{1}{2}\left(\frac{1}{T} - \frac{1}{C}\right)\left(\sigma_1 + \sigma_3\right) + \frac{1}{2}\left(\frac{1}{T} + \frac{1}{C}\right)\left(\sigma_1 - \sigma_3\right) \leq 1 \qquad (3)$$

The first term shows the normal stress effect and the second term the maximum shear stress effect. Relation (3) directly reduces to

$$\frac{\sigma_1}{T} - \frac{\sigma_3}{C} \leq 1 \qquad (4)$$

This remarkably simple form constitutes the entire two-property C-M failure criterion.

The C-M form was and is very appealing in its simplicity of concept and its generality and ease of use. Simple though it may seem, care still must be taken because it is often misinterpreted. There was great enthusiasm for it in the early 1900's when it came into general use. Then it was shown by von Karmen [6] and Böker [7] to have strong limitations for brittle materials, specifically natural minerals. Much effort has been expended over the years to "correct" and generalize this most simple and direct form of the C-M criterion, but to no special advantage or usefulness. Nevertheless it has continued to be widely used, perhaps because of the lack of a suitable two-property alternative.

In the limit of T=C the C-M form reduces to the maximum shear stress criterion of Tresca. In principal stress space this is given by the infinite cylinder of hexagonal cross section. When T≠ C the C-M form becomes a six-sided pyramid in principal stress space, and this six-sided pyramid has but 3-fold symmetry, rather than 6-fold symmetry.

An analytical form for this criterion in terms of invariants has been given by Schajer [8], although it is far easier to directly use the forms given here. The Coulomb-Mohr theory is used in applications as diverse as from nano-indentation to large scale geophysics.

As mentioned above, the C-M form is known to miss physical reality in a quantitative sense even though it has many of the proper ingredients in its formulation. Now after 232 years, this first developed theory of failure is still the standard by which a more suitable and comprehensive theory must be compared and judged. That is a special tribute to Coulomb and to Mohr.

Drucker-Prager Failure Theory

The Drucker-Prager [4] theory (D-P) probably was motivated by the desire for a general theory that would reduce to the Mises form in the limit, rather than the Tresca form. A course of action to accomplish this is quite direct. Simply replace the C-M six-sided pyramid in principal stress space by a circular cone.

Following this direction, write the possible failure criterion as

$$a\sigma_{ii} + b\sqrt{s_{ij}s_{ij}} \leq 1 \tag{5}$$

where s_{ij} is the deviatoric stress tensor involved in a Mises criterion, but now the term σ_{ii} explicitly brings in the mean normal stress effect which is implicit in the C-M formalism. Parameters a and b are to be determined such that (5) predicts uniaxial stress failure at the tensile value T and the compressive value C. It is easily shown that this then gives (5) as

$$\frac{1}{2}\left(\frac{1}{T} - \frac{1}{C}\right)\sigma_{ii} + \frac{1}{2}\left(\frac{1}{T} + \frac{1}{C}\right)\sqrt{\frac{3}{2}s_{ij}s_{ij}} \leq 1 \tag{6}$$

Finally, writing (6) out in terms of components gives

$$\frac{1}{2}\left(\frac{1}{T} - \frac{1}{C}\right)\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$
$$+ \frac{1}{2}\left(\frac{1}{T} + \frac{1}{C}\right)\left\{\frac{1}{2}\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2\right]\right.$$
$$\left. + 3\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\right\}^{1/2} \leq 1 \tag{7}$$

As shown, only the positive sign is associated with the square roots in (5)-(7).

In principal stress space Eq. (7) is that of a conical failure surface. When T=C the failure surface becomes the circular cylindrical form of Mises.

Like the C-M form the D-P form receives attention and usage, although not to as great an extent. For example, the D-P form was recently used by Wilson [9] in considering metal plasticity.

Present Failure Theory

Taken directly from Christensen [5], the present failure theory is given by the quadratic form

$$\alpha\left(\frac{\sigma_{ii}}{\kappa}\right) + \frac{3}{2}(1 + \alpha)\left(\frac{s_{ij}}{\kappa}\right)\left(\frac{s_{ij}}{\kappa}\right) \leq 1 \tag{8}$$

and if under the conditions of use and testing there is the condition $T/C \leq 1/2$ (giving the brittle range of behavior) then the following fracture criterion also applies

$$\sigma_1 \leq \sigma_{11}{}^T \quad \text{if} \quad \frac{\sigma_{11}{}^T}{\left|\sigma_{11}{}^C\right|} < \frac{1}{2} \tag{9}$$

where

$$\kappa = \left|\sigma_{11}{}^C\right| \tag{10}$$

$$\alpha = \frac{\left|\sigma_{11}{}^C\right|}{\sigma_{11}{}^T} - 1 \quad \text{for} \quad \alpha \geq 0 \tag{11}$$

and where σ_1 is the largest principal stress.

In component form and in the present notation (8) and (9) become

$$\left(\frac{1}{T} - \frac{1}{C}\right)\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$

$$+ \frac{1}{TC}\left\{\frac{1}{2}\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2\right]\right. \tag{12}$$

$$\left. + 3\left(\sigma_{12}{}^2 + \sigma_{23}{}^2 + \sigma_{31}{}^2\right)\right\} \leq 1$$

and

$$\sigma_1 \leq T \quad \text{if} \quad \frac{T}{C} < \frac{1}{2} \tag{13}$$

These forms are only applicable to materials for which $T/C \leq 1$.

Equation (12) is that of a paraboloid in principal stress space. The key difference between the conical form of (7) for D-P theory and the paraboloidal form of (12) for the present theory is due to the effect of the square root in (7) of the similar terms in each.

The fracture condition (13) provides the equations of three planes in principal stress space. At T/C = $\frac{1}{2}$ these planes are just tangent to the paraboloid. For T/C < 1/2, these fracture condition planes intersect the paraboloid giving three flattened sections in the otherwise full paraboloid. The size of these fracture cut off zones increases as T/C diminishes. The fracture condition (13) is not appropriate for use when 1/2 < T/C ≤ 1 because that corresponds to the ductile range of behavior. The non-dimensional parameter α in (8) and (11) provided the formalism needed to determine the ductile versus the brittle regions, which is necessary in order to prescribe the fracture criterion (13), see Christensen [5]. The use of the maximum principal stress in the fracture criterion (9) relates to the physical preference for the crack opening Mode I rather than Mode II in homogeneous materials, Christensen [10]. When T=C the entire criterion reverts to the Mises type circular cylinder in principal stress space.

The paraboloidal part of the present criterion, as with the pyramidal and conical forms, has a considerable history. It goes back to Sleicher [11], with interaction from von Mises [12]. Although it has received less attention than the other two forms, paraboloidal forms certainly have been examined, for example Stassi [13]. The complete formalism in (8)-(13) is a recent development, and the fracture restriction (9) in particular is a vital part of the present criterion in the brittle range.

Now the critical comparison of these three basic, two property failure theories can begin. All results in the following cases follow directly from the three failure formalisms just given.

Shear Stress, Eqi-Biaxial and Eqi-Triaxial Stresses

The simple shear stress, S, at failure from the three theories are found to be

$$S = \frac{TC}{T + C} \quad , \quad C\text{-}M \tag{14}$$

$$S = \frac{2TC}{\sqrt{3}\,(T + C)} \quad , \quad D\text{-}P \tag{15}$$

and

$$S = \frac{\sqrt{TC}}{\sqrt{3}} \quad \text{for} \quad \frac{T}{C} \geq \frac{1}{3} \left.\begin{array}{c} \\ \\ \end{array}\right\} \text{Present} \qquad (16a,b)$$

$$= T \qquad \text{for} \quad \frac{T}{C} \leq \frac{1}{3}$$

The values of the shear stress at failure from (14)-(16) at different values of T/C are shown in Table 1.

Although the differences between the C-M results and the other two are significant, there is nothing unusually extreme in any of these results. It may be noted that the value of S/C at T/C = 1/10 in the present theory would have been much larger than that in Table 1, were it not for the fracture condition (16b).

Turning now to states of eqi-biaxial stress, the tensile and compressive failure stresses for the three theories are found to be

$$\left.\begin{array}{l} \sigma_{2D}^T = T \\ \\ \sigma_{2D}^C = -C \end{array}\right\} C - M \qquad (17)$$

$$\left.\begin{array}{l} \dfrac{\sigma_{2D}^T}{T} = \dfrac{2}{3 - \dfrac{T}{C}} \\ \\ \dfrac{\sigma_{2D}^C}{C} = \dfrac{2}{\dfrac{C}{T} - 3} \end{array}\right\} D - P \qquad (18)$$

and

$$\left.\begin{array}{l} \dfrac{\sigma_{2D}^T}{T} = \left(1 - \dfrac{C}{T}\right) + \sqrt{1 - \dfrac{C}{T} + \dfrac{C^2}{T^2}} \\ \\ \dfrac{\sigma_{2D}^C}{C} = -\left(1 - \dfrac{T}{C}\right) - \sqrt{1 - \dfrac{T}{C} + \dfrac{T^2}{C^2}} \end{array}\right\} \text{Present} \qquad (19)$$

In states of eqi-triaxial tension the results are

$$\frac{\sigma_{3D}^T}{T} = \frac{1}{1 - \dfrac{T}{C}} \quad , \quad C\text{-}M \tag{20}$$

$$\frac{\sigma_{3D}^T}{T} = \frac{2/3}{1 - \dfrac{T}{C}} \quad , \quad D\text{-}P \tag{21}$$

$$\frac{\sigma_{3D}^T}{T} = \frac{1/3}{1 - \dfrac{T}{C}} \quad , \quad \text{Present} \tag{22}$$

In eqi-triaxial compression all three theories give the same result

$$\sigma_{3D}^C = -\infty \tag{23}$$

First, note from (20)-(22) for eqi-triaxial tension, the differences in failure stresses are in the ratios of 1:2:3 for the respective present, D-P and C-M theories irrespective of the values of T/C, and therefore occur in all ductile and brittle ranges. These are very large differences. It is not surprising that many schemes have been proposed for cutting off the apex regions of the C-M and D-P failure surfaces, Paul [14]. To do so, however, requires the insertion of more parameters into these theories, which then would need to be determined.

The eqi-biaxial and eqi-triaxial failure stress results are shown in Tables 2 and 3. Table 2 gives results for T/C = 1/3 which is somewhat into the brittle range while Table 3 is for the limit of completely damaged materials wherein the uniaxial tensile failure stress is negligible compared with the compressive value, T/C → 0. In general, the three methods give distinctly different results. The C-M method gives the eqi-biaxial results as being identical with the uniaxial results, the other two methods do not do so. The most unusual and unexpected result is the prediction from D-P theory that the eqi-biaxial compressive stress at failure is infinitely large at T/C = 1/3, Table 2. This value follows from σ_{2D}^C in Eq. (18) which ceases to be valid for values of T/C less than T/C = 1/3. The value of T/C = 1/3 is about that typical of cast iron. It is extremely unlikely that cast iron has an

unlimited resistance to eqi-biaxial compressive stress. This peculiar prediction from the D-P theory suggests looking more fully at biaxial stress states, as is done next.

General Biaxial Stress States

Consider a fully biaxial stress state involving σ_{11} and σ_{22} with all other stress components as vanishing. The C-M form follows directly from the forms given. The biaxial stress states for the D-P and present theories are given by

$$\sigma_{11}^2 + (C-T)\sigma_{11} - \frac{1}{4}\left(3\frac{C}{T} - 2 + 3\frac{T}{C}\right)\sigma_{11}\sigma_{22}$$
$$+ (C-T)\sigma_{22} + \sigma_{22}^2 = TC \ , \quad D\text{-}P \tag{24}$$

and

$$\sigma_{11}^2 + (C-T)\sigma_{11} - \sigma_{11}\sigma_{22} + (C-T)\sigma_{22} + \sigma_{22}^2 = TC \ , \quad \text{Present} \tag{25}$$

The fracture condition (13) must also be applied along with (25) in the present forms.

The corresponding failure forms are shown in Fig. 2 for the value T/C=1/3. If the value of T/C=1 were taken, the result would have been the Mises ellipse for the D-P and present forms and the six sided Tesca figure for the C-M case.

It is seen from Fig. 2 that the D-P theory gives an open ended form while the C-M and present forms give closed failure surfaces for T/C = 1/3. Furthermore, the C-M and present theories give closed failure forms for all values of T/C while the D-P forms are open for all T/C ≤ 1/3. This latter characteristic was discussed in the last section and found to be physically unrealistic. This is the first evidence of unrealistic behavior by any of the three failure theories, in this case the D-P theory. The mathematical reason for this behavior is straightforward.

The axis of the D-P cone in principal stress space makes equal angles with the three coordinate axes. At the value of T/C = 1/3 the cone has a sufficiently large opening angle such that the coordinate planes intersecting the cone produces curves of unlimited extent. This is the biaxial stress failure locus. A cone and a six-sided pyramid may seem quite similar, with the expectation that the C-M and D-P predictions would be only slightly different, similarly to the Mises and Tresca forms of the ductile limit. That reasoning however would be incorrect. For values of T/C considerably less than one, the six-sided

pyramid of C-M becomes closer to being a triangular pyramid, and the opening angles of the two forms are different. The results here reveal the two methods to be fundamentally different, and the C-M method is not subject to the same deficiency as that just found for the D-P method.

Shear Stress Plus Pressure

Shear stress by itself has been considered. Now take shear stress plus a superimposed pressure. Let τ be the shear stress and p be the pressure. The shear stress at failure is given by

$$\tau = \frac{TC}{T+C} + \frac{(C-T)}{(C+T)}p \quad , \quad C\text{-}M \tag{26}$$

$$\tau = \frac{2TC}{\sqrt{3}(T+C)} + \frac{\sqrt{3}(C-T)}{(C+T)}p \quad , \quad D\text{-}P \tag{27}$$

and

$$\tau = \sqrt{\frac{TC}{3} + (C-T)p} \quad \text{for } \frac{T}{C} \geq \frac{1}{3} \quad , \quad \text{Present} \tag{28}$$

In the present prediction, (28), for $\frac{T}{C} \leq \frac{1}{3}$, it is necessary to consider each case separately to determine whether the fracture criterion (13) applies instead of (28).

For the case of $\frac{T}{C} = \frac{1}{3}$ and $\frac{p}{C} = 4$ these results give

$$\left.\begin{array}{l} \dfrac{\tau}{C} = 2.25 \quad , \quad C\text{-}M \\[2ex] \dfrac{\tau}{C} = 3.75 \quad , \quad D\text{-}P \\[2ex] \text{and} \\[2ex] \dfrac{\tau}{C} = 1.67 \quad , \quad \text{Present} \end{array}\right\} \tag{29}$$

The differences of all three methods are considerable. For larger values of pressure, p, the differences between the present method and the other two become much larger, however, at some level of high pressure the predictions from all three methods must be

considered to be out of range. Although these differences are large, they cannot, by themselves be used to differentiate reasonable from unreasonable behavior for any of the three theories. The forms (26)-(28) and the results (29) do however show that the present failure methodology has a fundamentally different character from the other two.

Some Triaxial Stress States

Now some fully three-dimensional stress states will be considered which will show extreme differences between the three theories. As always, the C-M failure surfaces for any particular stress state follows directly from (3) or (4). The governing forms for the other two theories must be deduced from the forms given: D-P, Eq. (7) and present theory Eqs. (12) and (13).

To begin, two particular stress states will be considered

$$\text{i)} \quad \sigma_{11} \text{and} \, \sigma_{22} = \sigma_{33} \tag{30}$$

and

$$\text{ii)} \quad \sigma_{33} = \sigma_{11} \text{and} \, \sigma_{22} \tag{31}$$

The initial reason for choosing these two stress cases is that the C-M method is not able to distinguish between them. The other two methods will show a strong difference between the two cases.

For the D-P theory, the resulting failure envelopes are given by

D-P, Case i), σ_{11} and $\sigma_{22} = \sigma_{33}$

For
$$\frac{\sigma_{11}}{C} \geq \frac{2/3}{\dfrac{C}{T} - 1} \tag{32}$$

then

$$\frac{\sigma_{22}}{C} = \frac{2\left(\dfrac{T}{C} - \dfrac{\sigma_{11}}{C}\right)}{1 - 3\dfrac{T}{C}} \tag{33}$$

and for

$$\frac{\sigma_{11}}{C} \leq \frac{2/3}{\frac{C}{T} - 1} \tag{34}$$

then

$$\frac{\sigma_{22}}{C} = \frac{2\left(1 + \frac{\sigma_{11}}{C}\right)}{3\frac{C}{T} - 1} \tag{35}$$

Now for case ii) the results are found by the interchange of indices 11 and 22 in case i).

This is easily verified by the independent derivation of the forms for case ii).

<u>D-P</u>, Case ii), $\sigma_{33} = \sigma_{11}$ and σ_{22}

For

$$\frac{\sigma_{22}}{C} \geq \frac{2/3}{\frac{C}{T} - 1} \tag{36}$$

then

$$\frac{\sigma_{11}}{C} = \frac{2\left(\frac{T}{C} - \frac{\sigma_{22}}{C}\right)}{1 - 3\frac{T}{C}} \tag{37}$$

and for

$$\frac{\sigma_{22}}{C} \leq \frac{2/3}{\frac{C}{T} - 1} \tag{38}$$

$$\frac{\sigma_{11}}{C} = \frac{2\left(1 + \frac{\sigma_{22}}{C}\right)}{3\frac{T}{C} - 1} \tag{39}$$

In the present failure theory it is found that

<u>Present Theory</u>, case i), σ_{11} and $\sigma_{22} = \sigma_{33}$

$$\frac{\sigma_{11}}{C} = -\frac{1}{2}\left(1 - \frac{T}{C}\right) + \frac{\sigma_{22}}{C} \pm \sqrt{\frac{1}{4}\left(1 - \frac{T}{C}\right)^2 - 3\left(1 - \frac{T}{C}\right)\frac{\sigma_{22}}{C}} \tag{40}$$

Present Theory, case ii), $\sigma_{33} = \sigma_{11}$ and σ_{22}

$$\frac{\sigma_{22}}{C} = -\frac{1}{2}\left(1-\frac{T}{C}\right) + \frac{\sigma_{11}}{C} \pm \sqrt{\frac{1}{4}\left(1+\frac{T}{C}\right)^2 - 3\left(1-\frac{T}{C}\right)\frac{\sigma_{11}}{C}} \qquad (41)$$

The present theory also involves the fracture condition (13).

   With the C-M result from Fig. 1 and (4) and with (32)-(39) various comparisons can be made. The value T/C = 1/6 will be used as representing a quite brittle material such as a ceramic. The D-P, present and C-M results are as shown in Figs. 3, 4 and 5. In Figs. 3 and 4 the D-P and the present methods strongly distinguish between the stress states i) and ii) in (30) and (31). The C-M method, Fig. 5, cannot distinguish between these two cases, and this is a major shortcoming for the method. The reason the C-M shows this unusual behavior is that the method only uses the maximum and minimum principal stresses in its formulation. It cannot discriminate different cases involving wide fluctuations in the intermediate principal stress, as are involved in cases i) and ii). The two cases i) and ii) must be different when viewed in σ_{11}, σ_{22} space. Stress states near $\sigma_{11} = \sigma_{33}$ will be more tolerable than ones similarly near σ_{22} because of the greater stabilizing effect of the greater hydrostatic compressive stress for $\sigma_{11} = \sigma_{33}$ than for σ_{22}, when both are in the generally compressive region.

   The results from the three methods shown in Figs. 3-5 are very widely different from each other. In Fig. 3 it is seen that the D-P method gives an unusually widely opened form for the failure envelopes. The fracture cut off condition of the present theory is strongly apparent in Fig. 4. A much larger scale than that in Fig. 4 would be needed to see the detail in the tensile region near the origin.

   The examples just considered suggest looking at one further related example. Take a case of T/C = 1/10 which is still in the range of behavior for ceramics. Now take a stress state that is nearly that of uniaxial compression, but with a slight transverse pressure specified by

$$\frac{\sigma_{11}}{C} = -\beta$$

$$(42)$$

$$\frac{\sigma_{22}}{C} = \frac{\sigma_{33}}{C} = -\frac{\beta}{10}$$

The process now will be to solve for the value of β for failure from all three theories. If there were no transverse pressure, the value of β would simply be 1 for all three theories in uniaxial compression.

For the C-M criterion when stress state (42) is substituted into (4), it gives

$$(0) \beta \leq 1 \quad , \quad \text{C-M} \tag{43}$$

where the two terms in (4) cancel each other. For the D-P case when (42) is substituted into (7) the result is

$$\left(\frac{-9}{20}\right) \beta \leq 1 \quad , \quad \text{D-P} \tag{44}$$

Thus in both cases no matter how large is β, failure cannot occur.

Finally, using the stress state (42) in the present theory criterion (12), it is found that

$$\beta = \frac{2}{3} + \frac{\sqrt{46}}{9} \quad , \quad \text{Present} \tag{45}$$

To summarize then for a material with T/C=1/10, and for the nearly uniaxial stress state (42) the three theories predict

$$\text{C-M,} \quad \beta \to \infty \quad , \quad \text{no failure}$$

$$\text{D-P,} \quad \beta \to \infty \quad , \quad \text{no failure}$$

$$\text{Present} \quad , \quad \beta = 1.42 \quad , \quad \text{at failure}$$

The conclusion then is that for the stress state (42) representing only a slight deviation from a uniaxial compressive state, the C-M and D-P theories both predict a drastic shift in behavior actually giving a no failure result while the present theory predicts a failure value that is only slightly changed from that of uniaxial compression. This represents physically unrealistic behavior from both the C-M and D-P theories. Although this unrealistic behavior was demonstrated here for the value T/C = 1/10, the same behavior can be found for a wide range of T/C values. The reason for this abrupt transition from standard behavior to unacceptable behavior is that the stress state given by (42) lies entirely within the safety zone of both the C-M and D-P criteria, and the transition in

going from safety of stress state (42) to failure for uniaxial compression is extremely sensitive to the level of the transverse pressure in the cases of the C-M and D-P criteria.

Conclusions

Viewing the Coulomb-Mohr and Drucker-Prager forms as pyramidal and conical surfaces in principal stress space, they both are seen to be forms of first degree with respect to variations in the axial direction, starting at the apex. As shown here, both the C-M and D-P forms give some aspects of un-physical behavior, and in general both are inadequate for the purpose of failure characterization ranging from totally ductile to extremely brittle. The deficiency relates to the first degree form of the two criteria mentioned above resulting in a strong overestimation of tolerable stress in both the very compressive and the very tensile regions of principal stress space, along with other anomalies in the D-P case. For purposes of broad range applications and to correct these deficiencies, it appears necessary to characterize the failure function by a second-degree form (combined with a fracture failure mode). By no means is the present failure theory the only possible form of the type just specified. However, it does appear to be the only one that is presently available possessing these requisites and having general 3-D failure calibrated by just 2 properties for homogeneous and isotropic materials.

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Shear Stress, $\dfrac{S}{C}$, At Failure

	$\dfrac{T}{C} = 1$	$\dfrac{T}{C} = \dfrac{1}{2}$	$\dfrac{T}{C} = \dfrac{1}{10}$
C-M	1/2	0.333	0.091
D-P	$1 / \sqrt{3}$	0.385	0.105
Present	$1 / \sqrt{3}$	0.408	0.1

Table 1. Shear Stress At Failure

$$\text{Tension, } \frac{T}{C} = \frac{1}{3}$$

	$\dfrac{\sigma_{1D}}{T}$	$\dfrac{\sigma_{2D}}{T}$	$\dfrac{\sigma_{3D}}{T}$
C-M	1	1	$\dfrac{3}{2}$
D-P	1	$\dfrac{3}{4}$	1
Present	1	0.646	$\dfrac{1}{2}$

$$\text{Compression, } \frac{T}{C} = \frac{1}{3}$$

	$\dfrac{\sigma_{1D}}{C}$	$\dfrac{\sigma_{2D}}{C}$	$\dfrac{\sigma_{3D}}{C}$
C-M	-1	-1	$-\infty$
D-P	-1	$-\infty$	$-\infty$
Present	-1	-1.55	$-\infty$

Table 2. Eqi-Biaxial and Eqi-Triaxial Stresses, $\dfrac{T}{C} = \dfrac{1}{3}$

$$\text{Tension, } \frac{T}{C} \to 0$$

	$\dfrac{\sigma_{1D}}{T}$	$\dfrac{\sigma_{2D}}{T}$	$\dfrac{\sigma_{3D}}{T}$
C-M	1	1	1
D-P	1	$\dfrac{2}{3}$	$\dfrac{2}{3}$
Present	1	$\dfrac{1}{2}$	$\dfrac{1}{3}$

$$\text{Compression, } \frac{T}{C} \to 0$$

	$\dfrac{\sigma_{1D}}{C}$	$\dfrac{\sigma_{2D}}{C}$	$\dfrac{\sigma_{3D}}{C}$
C-M	-1	-1	$-\infty$
D-P	-1	$-\infty$	$-\infty$
Present	-1	-2	$-\infty$

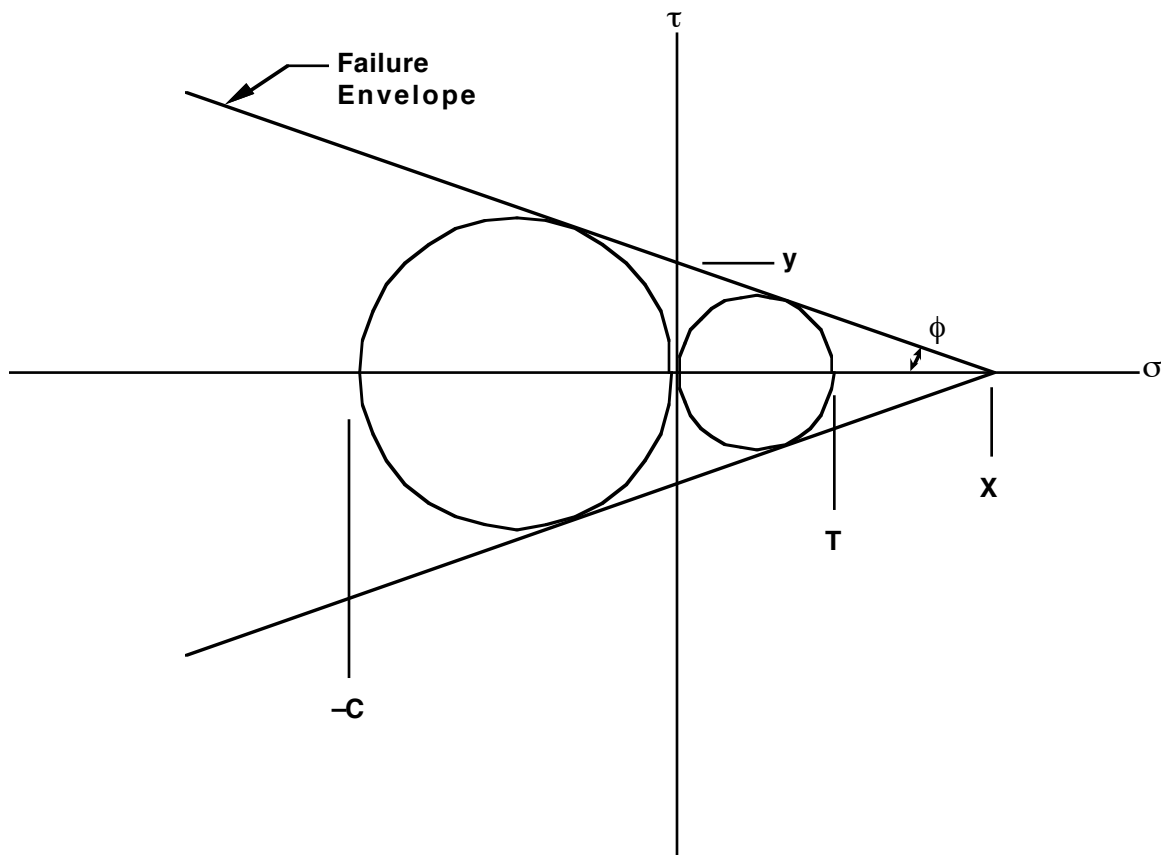Table 3. Eqi-Biaxial and Eqi-Triaxial Stresses, $\dfrac{T}{C} \to 0$

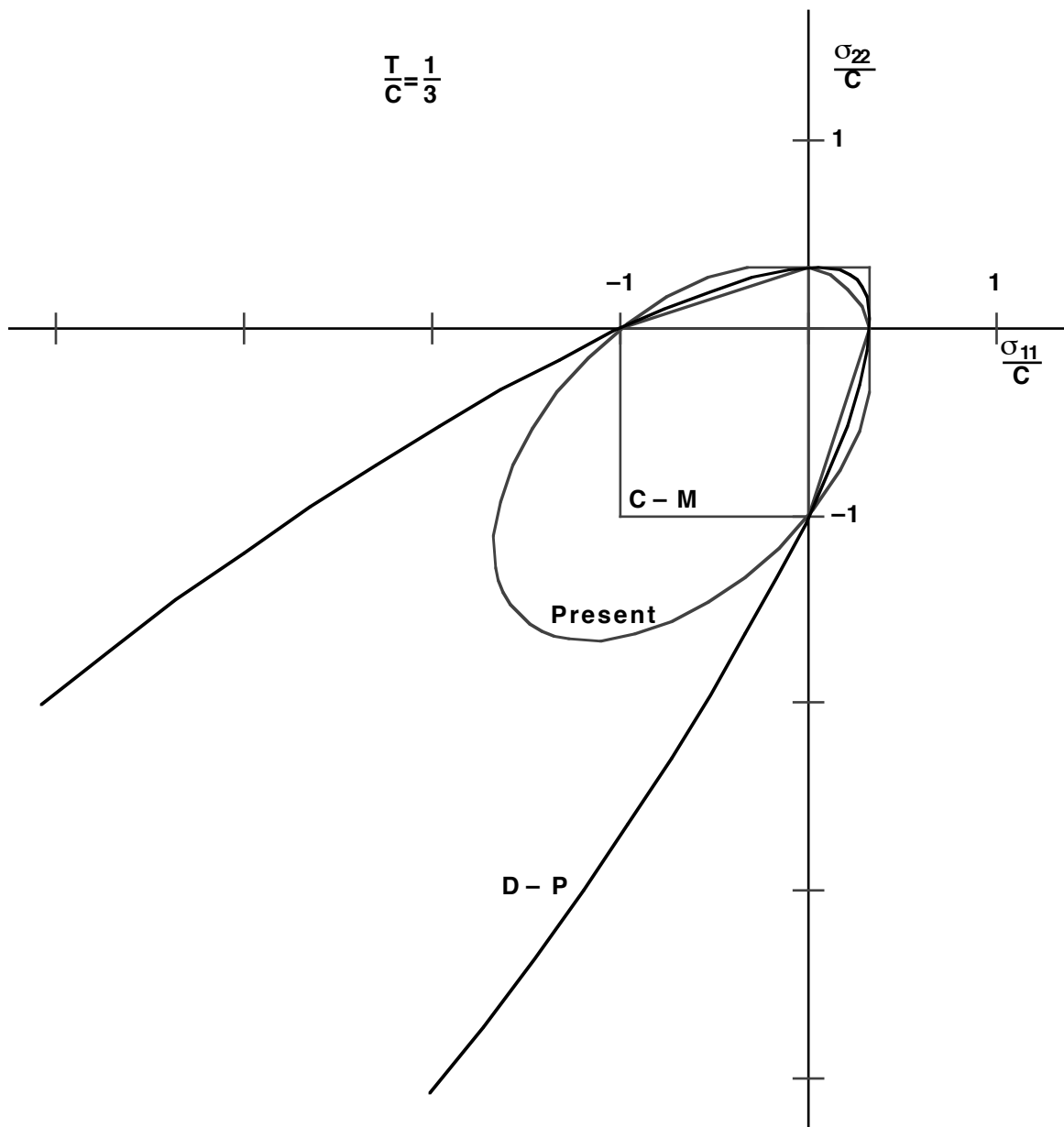Figure 1. Coulomb-Mohr Failure Form

Figure 2. Biaxial Stress State Failure, T/C =1/3

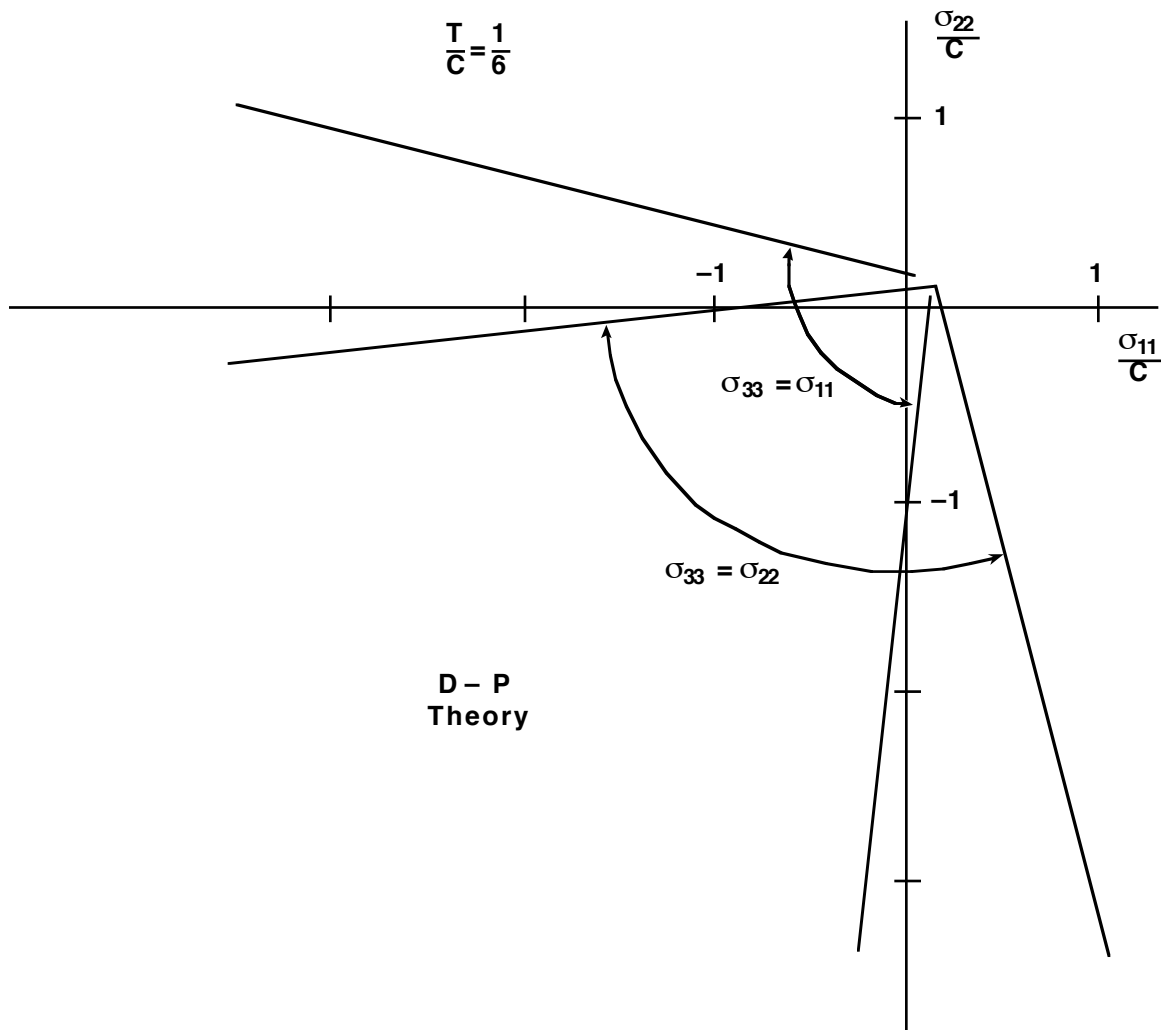Figure 3. Stress States σ_{11} vs $\sigma_{22} = \sigma_{33}$ and $\sigma_{11} = \sigma_{33}$ vs σ_{22}, D-P Theory
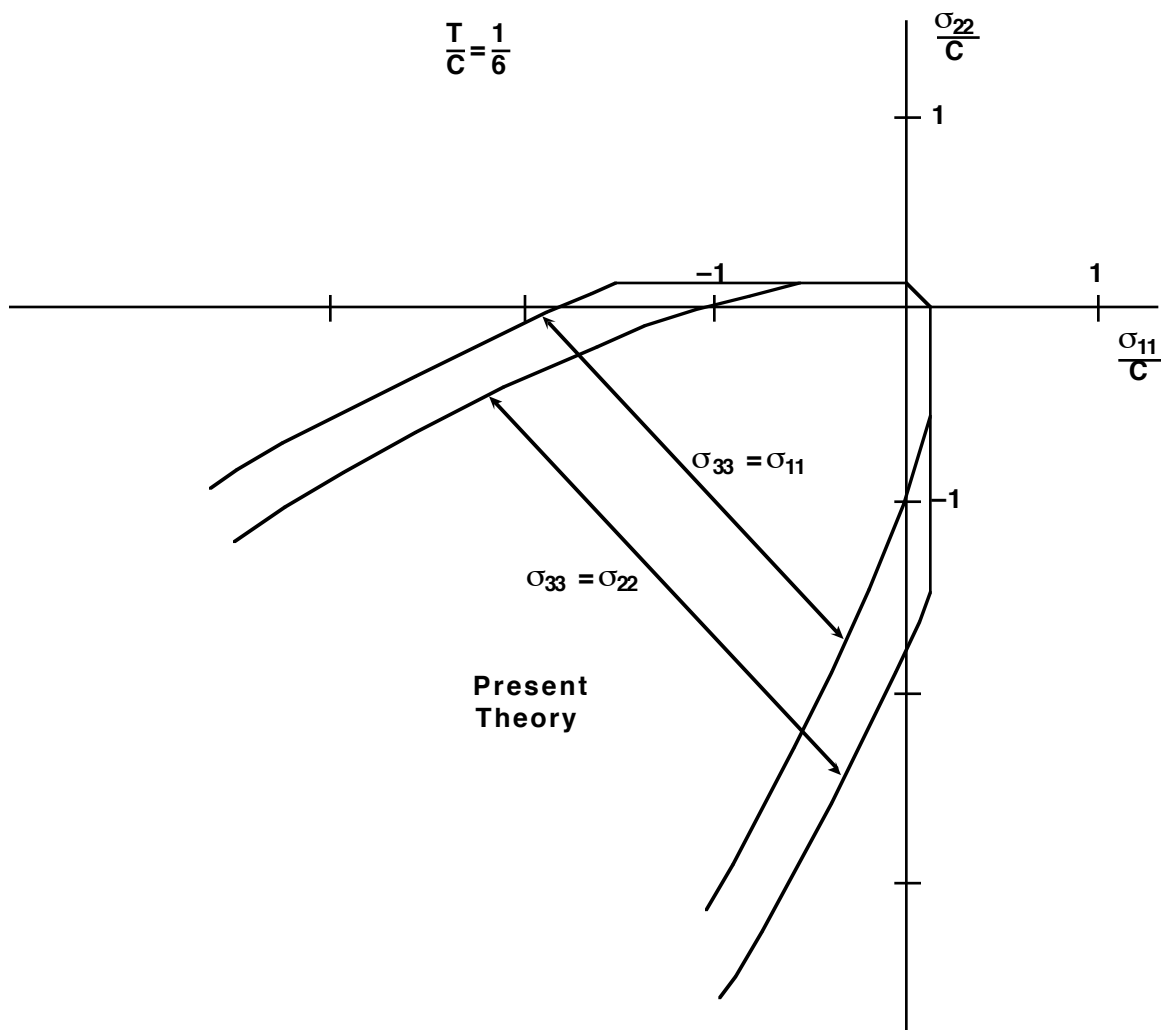
Figure 4. Stress States σ_{11} vs $\sigma_{22} = \sigma_{33}$ and $\sigma_{11} = \sigma_{33}$ vs σ_{22}, Present Theory
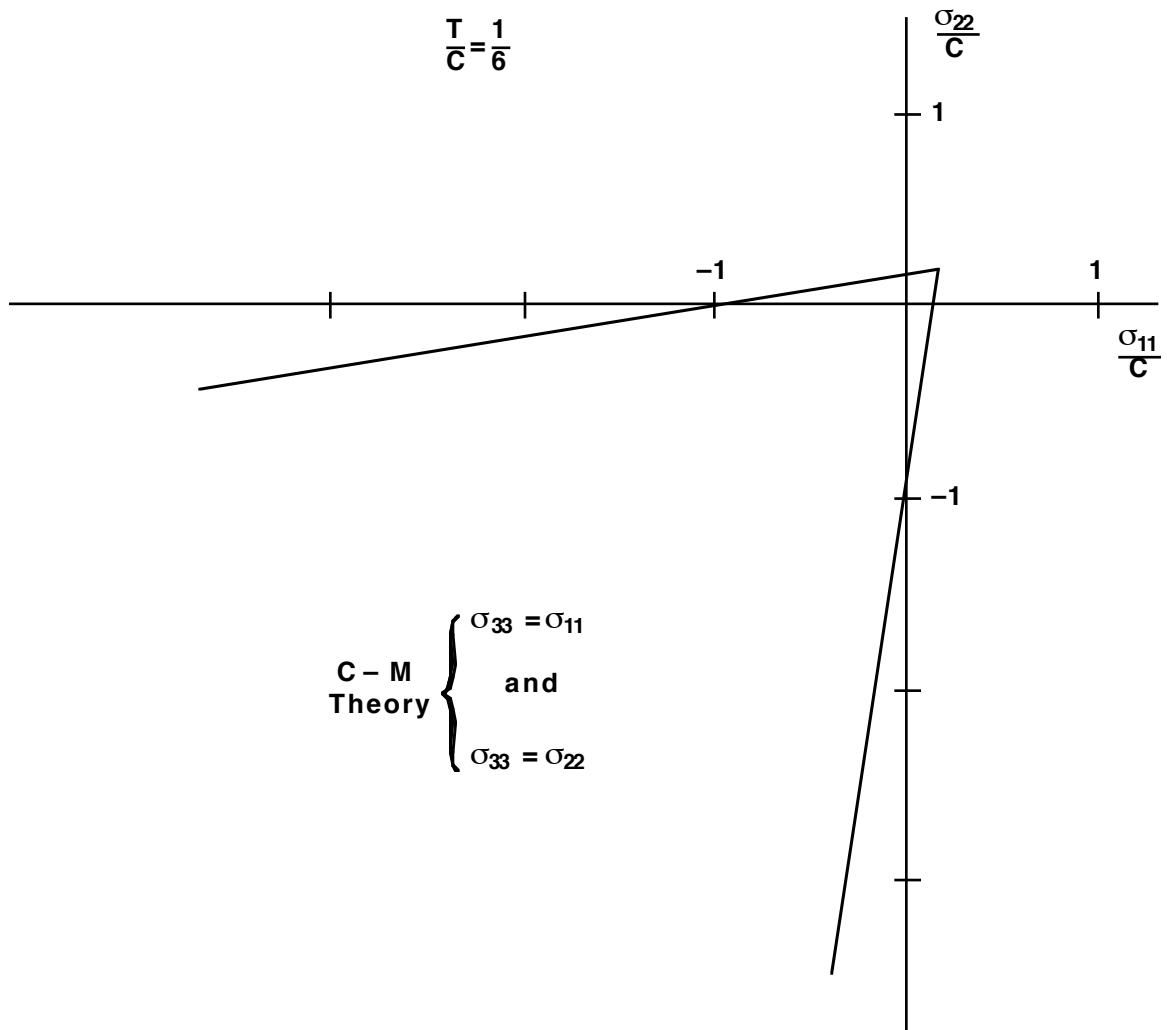
$$\frac{T}{C} = \frac{1}{6}$$

$$\frac{\sigma_{22}}{C}$$

1

−1

1

$$\frac{\sigma_{11}}{C}$$

−1

$$\text{C − M} \atop \text{Theory} \left\{ \begin{array}{c} \sigma_{33} = \sigma_{11} \\ \text{and} \\ \sigma_{33} = \sigma_{22} \end{array} \right.$$

Figure 5. Stress States σ_{11} vs $\sigma_{22} = \sigma_{33}$ and $\sigma_{11} = \sigma_{33}$ vs σ_{22}, C-M Theory