A Two Property Yield, Failure (Fracture) Criterion for Homogeneous, Isotropic Materials

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Abstract

A critical review is given of the various historical attempts to formulate a general, three-dimensional theory of failure for broad classes of homogeneous, isotropic elastic materials. Following that, a recently developed two-parameter yield/failure criterion is compared with the historical efforts and it is further interpreted and extended. Specifically, the yield/failure criterion is combined with a fracture restriction that places limits on certain tensile stress states, without involving any additional parameters. An evaluation is conducted using available experimental data obtained from a variety of materials types. The two materials parameters are given a primary designation as yield type properties over a specified range of ductile behavior, and as failure or fracture type properties over the complementary brittle range.

Historical Review

Solid materials have long posed special problems for physical characterization, whether they be of amorphous type or crystalline. This is especially true for the characteristic of physical, mechanical failure, which is the subject of interest here.

It is technically helpful and surprisingly revealing to examine the history for this field of materials failure. Many of the great personages of science were intimately involved. A brief account will now be given of the major historical turning points since they are of importance in assessing the current state of knowledge and practice in this field. From this point on, in the historical account and in the work to follow, the term failure or failure characterization will be used in an inclusive sense as applying to either or both of the conditions of yielding or actual failure. At the end of this article in the Conclusions section, criteria will be specified as to which conditions pertain to initial yield and which to explicit failure behaviors. All consideration here are with homogeneous isotropic materials, where the historical goal was to find a general failure criterion that would cover the full range from ductile to brittle types of materials.

Probably one of the most misunderstood failure concepts is that of the Coulomb-Mohr type. Its antecedents go back to nearly the beginning of all of mechanics and its popular reception continues up to the present day. It comes from the original conception of Coulomb (1773) that on a failure surface the shear stress τ at failure is related to the normal stress, σ, acting across the failure surface as

$$\tau \leq c - \mu \sigma$$

where the two parameters c and μ are material specific. With $\mu = 0$ this is just the maximum shear stress criterion usually known as the Tresca form today. Coulomb was a

brilliant engineer working at military installations in early adulthood, much concerned with structures and stability, possibly of soil embankments. In fact, he labeled parameter μ as the coefficient of internal friction, suggestive of granular material flow or incipient flow. This simplest form relates shear stress on an assumed failure plane to the transverse normal stress (tensile or compressive) as shown in Fig. 1. Thus, the failure criterion is assumed to depend only upon the tractions acting upon the failure surface and it is independent of the stress components in the plane of the assumed failure surface. The two parameters, c and μ, determine the intercepts and the slope of the straight line failure envelopes in Fig. 1.

Mohr (1900) recognized that the straight line envelopes in Fig. 1 would not be likely and that the envelopes in general would be curved, concave inward. Mohr also gave an interpretation especially applicable to Coulomb's two parameter criterion. The maximum and minimum principal stresses are used to form a "Mohr's circle" construction within the failure envelope of Fig. 1. Incipient failure occurs when the circle has tangency with the envelope as illustrated for uniaxial tension and compression in Fig. 1. The uniaxial tensile and compressive failure stresses can be used to calibrate the two failure parameters. This criterion and procedure usually is known now as the Coulomb-Mohr form. For brittle materials the Coulomb-Mohr method predicts that uniaxial and eqi-biaxial tensile failure are at the same levels, but that eqi-triaxial tensile failure is much stronger. This latter unphysical behavior has no supporting evidence and likely only occurs in the ductile range. In testing geological materials, von Karman (1912) and also Böker (1915) showed that the Coulomb-Mohr criterion is more successful when the stresses are somewhat compressive and less successful otherwise, either in the very compressive or slightly compressive region, the latter of which is unfortunately the most limiting region.

With all these difficulties understood, more involved forms of the Coulomb-Mohr type can be constructed with curved envelopes or connecting segments or caps replacing the straight lines of Fig. 1. This procedure then involves introducing more parameters than just the two involved with the simplest form. The procedure quickly becomes quite complex. Nor is there anything particularly fundamental about the Coulomb-Mohr approach, as is often claimed. It is simply a well developed hypothesis about failure, but it is not unique, and not necessarily useful unless proven to be so. Examples will be given later further showing its deficiencies.

None of this detracts from the original contribution of Coulomb, nor of Mohr for that matter. Coulomb's recognition that the shear stress at failure would be influenced by the transverse normal stress is absolutely correct, and his form is of considerable application in soil mechanics and rock mechanics over a limited range. The useful application of the Coulomb-Mohr type to materials science and technology has been embryonic and never gotten beyond that stage.

It was Maxwell who had the acuity and perception to first see the use of energy as a possible or partial failure criterion. In correspondence with Lord Kelvin in 1856, Maxwell anticipated what is now known as the Mises criterion by saying, "I have strong reasons for believing that when the strain energy of distortion reaches a certain limit then the element will begin to give way" (Timoshenko (1953)). Much later Huber (1904) and still later Mises (1913) gave this the form in which it is used today for the yielding of

ductile metals. Over the years the apparent appeal of various forms of energy as a general failure criterion has been elusively strong. Beltrami (1889) proposed the total energy, energy of distortion plus energy of volume change, be used as the general criterion, but this is clearly contradicted by the ductile metals case, and other materials as well. Only the Mises case for distortional energy is indisputably correct when applied to ductile metals, but not for any other materials type. G. I. Taylor, Taylor and Quiney (1931), gave the conclusive experimental verification for this ductile metals criterion.

In addition to other approaches, maximum normal stress and maximum normal strain failure criteria had distinguished and strong adherents, Lamé for the former and Saint-Venant for the latter (Timoshenko (1953)). Neither came into a state of explicit validation and meaningful application although they certainly added to the historical state of confusion. These two criteria are sometimes still used, with no apparent justification.

It should not be inferred that the historical efforts were not extremely helpful. As already implied, the profound deductions of Coulomb and of Maxwell were the most important steps of all time in this field. Perhaps each was motivated by a particular materials behavior at opposite ends of the scale. Other scientists certainly understood the essential ingredients of the problem. For example, as discussed by Todhunter and Pearson (1893), Lord Kelvin held the view that "It may be inconceivable that any amount of uniform pressure applied to the surface of a solid sphere of isotropic material would cause it to rupture,but it is also very difficult to believe that a uniform tension, if it could be applied to its surface, would not, were it indefinitely increased, produce rupture". Thomson and Tait (1879). However, that understanding did not lead to progress toward a failure criterion for application. Despite more than a century of intense inquiry, Love (1927) would write, "The conditions of rupture are but vaguely understood". The term rupture had about the same implied meaning as failure or fracture would today.

After the many attempts to cast the Coulomb-Mohr hypothesis into a generally realistic, accessible form never came to fruition, and the quest for a general energy criterion necessarily came to an unsuccessful end, a degree of pessimism seemed to influence the further efforts to find a general criterion. Thereafter, the attempts were direct postulations of particular forms, appealing on some basis to each originator, but seemingly done with a motivation to just see what would happen. None of the general approaches in the modern era had a solid basis in a physical derivation along with a critical examination, there have only been demonstrations. For some examples see the survey by Paul (1968).

The only related significant development in modern times has been the remarkable emergence of fracture mechanics. It has indeed been a tremendous development with far reaching implications. Griffith (1921) is widely and justifiably credited with originating fracture mechanics. In lesser known work, Griffith (1924) also went a step further and adapted the crack induced failure effect into homogeneous material behavior and thereby prescribed a failure criterion with a universal ratio of uniaxial compressive to tensile failure stresses at exactly the value of 8. Murrell (1963) extended this Griffith treatment and derived the compressive to tensile ratio as being exactly 12. Neither result has been found to have general applicability. The relationship, if any, of general criteria of the type sought here to fracture mechanics has remained clouded in this more modern era.

Much more will be said later in this work of the relationship between fracture behavior and failure criteria for homogeneous materials.

Coming up to the present times, clearly the state of general three dimensional failure characterization and criteria is not satisfactory. The long and tortuous history of the subject must be due to the scale of the underlying difficulties. Perhaps it is not possible to formulate a general failure criterion. This would mean that the problem will always need to be approached on an empirical, case by case basis for the many different classes and types of materials. However, before capitulating to the perceived and pervasive complexity, there remains the possibility that the expanding knowledge base of the modern era may provide the advantage needed for a more successful approach. Some very recent work does in fact offer a new view of this field. In the following body of this work, some results from this newly developed theory of failure, Christensen (1997), will be examined, extended and evaluated. Other new approaches may also be expected, at least if the enduring importance of failure characterization gives any indication.

Yield/Failure Criterion

Although the central purpose here concerns failure characterization, it is useful to start with the elastic energy representation and with the most basic source from which it is obtained. Ultimately a type of bridge will be established between the elastic energy and the failure forms.

For three dimensionally isotropic materials, take a polynomial expansion for any scalor, physical quantity of interest in terms of stress, σ_{ij}, One specific form, given most compactly in tensor notation, is

$$L = \Delta\, \sigma_{ii} + \zeta\, \sigma_{ii}^{2} + \eta\, s_{ij}\, s_{ij} \tag{1}$$

Where Δ, ζ, and η are the specifiable parameters, σ_{ii} is the dilatational stress and the deviatoric (shear) stress is given by

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk} \tag{2}$$

The expansion (1) is one of several different but equivalent forms of the most general polynomial expansion of second degree for an isotropic representation in terms of stress.

Now consider the elastic energy U. For a material with an unloaded preferred configuration, the requirement that the elastic energy be positive definite requires the parameter Δ in (1) to be taken as vanishing. Then the energy form is given by

$$U = \frac{1}{2E}\left[\frac{(1-2\nu)}{3}\sigma_{ii}^{2} + (1+\upsilon)s_{ij}s_{ij}\right] \tag{3}$$

and where E is the Young's modulus and ν is Poisson's ratio with $-1 \le \nu \le 1/2$. It is advantageous to write (3) in a slightly more transparent form for later comparison with the failure criterion. Take

$$\beta = \frac{(1 - 2\upsilon)}{3} \tag{4}$$

With (4), the elastic energy (3) becomes

$$U = \frac{E}{2}\left[\beta\left(\frac{\sigma_{ii}}{E}\right)^2 + \frac{3}{2}(1-\beta)\left(\frac{s_{ij}}{E}\right)\left(\frac{s_{ij}}{E}\right)\right], \qquad 0 \le \beta \le 1 \tag{5}$$

The energy form, (3) and (5), involving E and ν or β is one of the basic forms in which the elastic energy can be written. The property E can be seen as a scaling amplitude (1/E) while the non-dimensional property ν or β specifies the degree of coupling or interaction between the dilatational and the deviatoric energy terms each varying from no contribution to the fully dominant contribution, over the range $0 \le \beta \le 1$. There is no involvement in (5) with properties that must become unbounded at the β value limits of behavior, such as with the shear modulus or the bulk modulus.

It is now hypothesized that the fundamental stress state descriptors used for characterizing the elastic energy are also likely to be the best stress state descriptors that could be used for characterizing the limit of the elastic energy range, beyond which irreversible deformation and damage occurs. In fact, it would appear that the method to be followed here is one of the few internally consistent approaches for a general treatment of both the elastic energy representation and the limiting range of the elastic energy representation (failure). This method should not be confused with simply using the elastic energy itself as the failure criterion, rather, the same source for each is being used.

Following the procedure just specified, take the polynomial expansion (1) as also controlling the failure form in terms of stress. Impose the physical assumption that the homogeneous material does not fail under a state of hydrostatic pressure, although it could fail under hydrostatic tension. The plausibility of this condition goes back at least as far as to Kelvin. This assumption or condition requires that the ζ parameter in (1) be taken as vanishing. The resulting failure criterion directly from (1) is shown by Christensen (1997) to take the form

$$\alpha\left(\frac{\sigma_{ii}}{\kappa}\right) + \frac{3}{2}(1+\alpha)\left(\frac{s_{ij}}{\kappa}\right)\left(\frac{s_{ij}}{\kappa}\right) \le 1 \tag{6}$$

where the two failure parameters α and κ are given by

$$\kappa = |\sigma_{11}{}^{C}| \tag{7}$$

and

$$\alpha = \frac{\left|\sigma^{C}_{11}\right|}{\sigma^{T}_{11}} - 1 \tag{8}$$

for the class of homogeneous materials for which

$$0 \le \frac{\sigma^{T}_{11}}{\left|\sigma^{C}_{11}\right|} \le 1, \qquad \alpha \ge 0 \tag{9}$$

In (6)-(9) σ_{11}^{T} and σ_{11}^{C} are the uniaxial stress state yield or failure values in tension and compression. Parameter α is of nondimensional form and it presently will be shown to cover the range from very ductile behavior at $\alpha = 0$ to very brittle behavior as $\alpha \rightarrow \infty$. In obtaining (6) from the form in Christensen (1997) the small notational change of $\kappa = \sqrt{3}\,k$ has been made which interestingly brings the failure form (6) into symbolic alignment with the elastic energy form (5). The full failure criterion (6)-(8) is written out in terms of components in the Evaluation section. Form (6) is stated as an inequality or equality to represent both the elastic range and its limit, the failure criterion.

The failure form (6) certainly is not positive definite, but it has a special relationship with the elastic energy form (5) due to their derivation from the common source. These two forms have essential difference only in the way in which the dilatational stress term enters each, one linearly and one quadratically. At $\alpha = \beta = 0$ both forms eliminate the dilatational stress effect. The special relationship between the failure form and the elastic energy form will be further discussed in the last section. In the remainder of this section some previously derived results will be briefly summarized for later use.

For uniaxial stress, σ_{11}, the energy form (5) becomes

$$U = \frac{1}{2E}\left[\beta\sigma_{11}^{2} + (1-\beta)\sigma_{11}^{2}\right] \tag{10}$$

which shows how relation (5) proportions the energy between the dilatational and distortional sources for any stress state when ν and thereby β are specified. Again for uniaxial stress, σ_{11}, the corresponding failure form (6) becomes

$$\alpha\left(\frac{\sigma_{11}}{\kappa}\right) + (1+\alpha)\left(\frac{\sigma_{11}}{\kappa}\right)^{2} = 1 \tag{11}$$

Solving (11) for the two roots of σ_{11} gives

$$\sigma_{11}^{C} = -\kappa \tag{12}$$

and

$$\sigma_{11}^{T} = \frac{\kappa}{1+\alpha} \tag{13}$$

Now solving (12) and (13) for κ and α gives (7) and (8).

Next, consider the limiting cases of the failure criterion (6). At $\alpha = 0$ (6) becomes

$$\frac{1}{2} s_{ij} s_{ij} \leq \frac{\kappa^2}{3}, \qquad \alpha = 0 \tag{14}$$

which is just the usual Mises criterion. Alternatively, letting α become very large gives from (6)

$$\frac{1}{2} s_{ij} s_{ij} \leq -\frac{\kappa \sigma_{ii}}{3}, \qquad \alpha \to \infty \tag{15}$$

For shear stress τ and pressure p relation (15) is simply

$$\tau^2 \leq |\sigma_{11}^{C}|\, p \tag{16}$$

These last two relations show that sufficient pressure can be applied to give the material a Mises-like character even in this extreme brittle condition of $\alpha \to \infty$. Another limiting case type behavior will be given in the Fracture Restriction section.

In order to investigate a ductile-brittle transition type behavior, the effect of a hydrostatic stress state as a "pre-stress" condition has been considered. In a rather complex derivation, Christensen (2000), involving the concept of an "intrinsic strength" it was found that the effect of a pressure type stress upon the material is equivalent to that of a change in the value of α in (6). It was specifically found that at $\alpha = 1$

$$\left.\frac{d\alpha}{dp}\right|_{\alpha=1} = -\frac{1}{\sigma_{11}^{T}} \tag{17}$$

where p is the pressure. Thus, increasing the pressure decreases the value of an equivalent α, moving the behavior toward $\alpha = 0$, the ductile Mises end of the scale. In general the derivative $d\alpha/dp$ involves the specified stress state through σ_{ii} and $s_{ij}s_{ij}$ but only at $\alpha = 1$ do these terms disappear and the behavior shown in (17) emerges. Relation (17) was interpreted as that of a property of the material. It was further interpreted by Christensen (2000) that this behavior at $\alpha = 1$ is that of the much studied transition between ductile and brittle behaviors, thus

$$\frac{1}{2} < \frac{\sigma_{11}^{T}}{\left|\sigma_{11}^{C}\right|} \leq 1 \qquad \text{Ductile Behavior } (\alpha < 1) \qquad (18)$$

$$\frac{\sigma_{11}^{T}}{\left|\sigma_{11}^{C}\right|} = \frac{1}{2} \qquad \text{Ductile-Brittle Transition } (\alpha = 1) \qquad (19)$$

$$0 \leq \frac{\sigma_{11}^{T}}{\left|\sigma_{11}^{C}\right|} < \frac{1}{2} \qquad \text{Brittle Behavior } (\alpha > 1) \qquad (20)$$

This transition at $\alpha = 1$ is not an abrupt change from fundamentally and completely ductile behavior into a totally brittle behavior. There is instead a complete spectrum of gradually changing behaviors with changes in α. Nevertheless, as indicated by (17) and as will be seen in the next section there is a distinctive behavioral feature at $\alpha = 1$ that marks the incremental shift in general failure characteristics.

In the ductile range of behavior and for deformation beyond the initial yield surface, letting parameter κ depend upon the proper field variables would give isotropic hardening. Alternatively, letting only α depend on the proper field variables would give a type of kinematic hardening.

Fracture Restriction

The failure form (6), although appearing simple, contains a wide range of behaviors as already seen from the limiting cases and will be seen from the fracture considerations to follow. It is unlikely that this approach would have been forseen before fracture mechanics was developed and its significance assimilated. It is particularly important that this criterion be fully integrated with fracture effects, as will now be approached, otherwise it would be incomplete and of uncertain orientation.

Following Christensen (1997) take the situation of a normal stress, say σ_{11}, with the added effect of a small increment of another transverse stress, say σ_{22}. Using the failure criterion (6) it follows in a few steps that at $\alpha = 1$ and under tension there is no effect of the small increment of superimposed stress, σ_{22}. This result is expressed as

$$\left.\frac{d\sigma_{11}^{T}}{d\sigma_{22}}\right|_{\sigma_{22}=0} = 0 \quad \text{at} \quad \alpha = 1 \qquad (21)$$

It is only at $\alpha = 1$ that this derivative vanishes and only for tension, not compression. Consistent with fracture mechanics, the form (21) admits the interpretation as being due to a crack opening stress, at $\alpha = 1$, and not dependent on the other stress components so

long as they are small. This behavior means that for σ_{22} small compared with σ_{11} the crack with the most damaging orientation relative to σ_{11} is activated and the effect of small σ_{22} is completely negligible, as with classical fracture mechanics. When σ_{22} is not small compared with σ_{11} then the randomly oriented cracks are more likely to be activated or interact and degrade the strength still further, in tensile states. Of course this described relationship to classical fracture mechanics is quite idealized, but still is useful here as providing an interpretive and guiding directive. The final proof of any predictive behavior must come through comparison with reliable experimental data, as will be considered in the next section.

Relation (21) is the first appearance here of a fracture type behavior with increasing values of α. This behavior is implicit in the preceding criterion (6)-(8). Now, a further fracture restriction will be imposed which is external to the preceding criterion, but closely related to it.

It can be shown that the second derivative corresponding to (21) is negative, thus σ_{11}^T is a local maximum at $\alpha = 1$. More generally, at $\alpha = 1$, it can be shown that σ_{11}^T is the largest possible principal stress for any stress state. This directly associates the above shown fracture characteristic, (21), with the largest principal stress, at $\alpha = 1$. Still more generally, for the brittle range of behavior a fracture restriction will be specified corresponding to this behavior at $\alpha = 1$. For other values of α besides that of $\alpha = 1$, a normal stress again, say σ_{11}, could be larger than σ_{11}^T when evaluated from (6) –(8). However, it is expected that this would not be physically possible for brittle materials because of the intervention of fracture. A basic premise of fracture behavior would be that no component of normal stress could be larger than σ_{11}^T, for the range of brittle materials. This premise will now be implemented as the fracture restriction that must be included as part of the overall failure criterion, along with (6)-(8). The brittle range of behavior is given by (20). The fracture restriction is accordingly given by

$$\sigma_1 \leq \sigma_{11}^T \quad \text{if } \alpha > 1 \tag{22a}$$

where σ_1 is the largest principal stress. Alternatively express (22a) in terms of the α and κ failure parameters using (13) as

$$\sigma_1 \leq \frac{\kappa}{1+\alpha} \quad \text{if} \quad \alpha > 1 \tag{22b}$$

It is important to recognize that this fracture restriction, along with (6)-(8) does not impose any new failure parameters. It is simply a statement of the basic nature of fracture intervention from the competing fracture mode under certain stress states for brittle materials. The fracture restriction is considered to be basic to the present failure theory, which still remains at the two parameter level.

The effect of the fracture restriction (22) is to provide a "fracture cutoff' for the forms from (6)-(8) in certain subspaces of stress for materials in the brittle range of behavior (but not in the ductile range). The procedure is not complicated, simply use both criteria, (6)-(8) and (22), and always take the most restrictive region as that allowed within the failure envelope. An example will help further to explain the rationale and the procedure.

Take the particular case of

$$\frac{\sigma_{11}^{T}}{\left|\sigma_{11}^{C}\right|} = \frac{1}{3} \quad , \quad \alpha = 2$$

Materials with a tensile failure stress one-third the value of the compressive level would be considered to be quite brittle. Common cast iron is an example. For $\alpha = 2$ it follows from failure criterion (6) that the failure stress in simple shear is the same as that in simple tension,

$$\sigma_{12}^{Y} = \sigma_{11}^{T}$$

In a biaxial stress state with σ_{11} and σ_{22}, the failure function (6) is as shown by the smooth curve in Fig. 2. The fracture cutoff from criterion (22) is shown by the two straight line segments in Fig. 2. Consider the second quadrant in Fig. 2 wherein the point at $\sigma_{22} = -\sigma_{11}$ is that of simple shear. One can view the mechanism of brittle failure due to shear stress as being induced by fracture from the rotated tensile component as is observed from the failure plane orientation in the testing of brittle materials. This fracture surface orientation is shown in Fig. 3 for a cast iron specimen tested in torsion, Popov (1990), as often displayed in introductory texts on mechanics of materials. Moving away from the point $\sigma_{22} = -\sigma_{11}$ in the second quadrant of Fig. 2, it is physically unlikely that by reducing the magnitude of the orthogonal compressive stress component, σ_{11}, that the tensile stress, σ_{22}. can be made to increase, under these fracture conditions. This unrealistic, hypothetical behavior is precluded by the fracture cutoff in the brittle range. Figure 2 reveals how limited the brittle, tensile stress sensitive region can be relative to the overall yield/failure locus.

An example at $\alpha = 1$ would show that the size of the fracture cutoff zone diminishes to a point at that value of α. Thus the value of $\alpha = 1$, described in the preceding section as that of the ductile-brittle transition, is also the value of α at which the fracture intervention effect first emerges through the fracture restriction (22).

It should be mentioned that the Coulomb-Mohr form can be extended to include a specific fracture intervention effect by including more parameters than just the two in the basic Coulomb-Mohr form. For this and many other mostly fracture purpose forms, the rock mechanics literature should be consulted, Jaeger and Cook (1979). Notable among such efforts is the frictional fracture model of McClintock and Walsh (1962).

Now move to the situation of parameter α being very large, the extreme brittle case. This extreme range of behavior was considered in Christensen and DeTeresa (2001), as outlined now. Expand the stress components in powers of $(1/\alpha)$ and substitute into the failure criterion (6). Collecting powers of $(1/\alpha)$ and retaining only lowest order terms gives two separate results depending upon the sign of the mean normal stress. For negative mean normal stress the result already stated as (15) is determined, showing how sufficient pressure induces a Mises-like behavior. But for positive mean normal stress an

unusual and distinctive result emerges. It is found that the failure criterion in the limit becomes

$$\sigma_{ii} \leq \sigma_{11}{}^{T} \qquad \text{for } \sigma_{ii} > 0\,, \qquad \alpha \rightarrow \infty \tag{23}$$

controlled by positive mean normal stress.

The situation with α very large is that the material is so damaged by a state of high crack density or by a micro network of weak interfaces or by other similar physically degrading effects that the behavior is very different depending on the sign of the mean normal stress. For compressive mean normal stress, the effect of the pressure is to give the material a weakly Mises-like behavior as already mentioned. But when the mean normal stress is positive the most favorable failure mode is that involving the mean normal stress itself, (23), as a crack opening fracture mode. These limiting case results are helpful in understanding the complexities of the extreme brittle case. However, it would not be advisable to use the asymptotic results (15) or (23) as failure criteria directly when the more general form (6) with no restrictive assumptions of limit case behavior is available. Furthermore, these asymptotic results (15) and (23) are subject to the fracture restriction (22).

The overall fracture behavior is well illustrated by the example of Fig. 2. In the first quadrant of Fig. 2 and the straight line portions of the 2nd and 4th quadrants, the failure type would be considered to be that of brittle fracture. The first quadrant form is that of brittle behavior as increasingly related to positive mean normal stress.

The general outline of homogeneous material behavior is now becoming more evident within the present context. At one extreme, $\alpha = 0$, the behavior is that of a completely ductile type governed by the Mises criterion. At the other extreme, $\alpha \rightarrow \infty$, for the mean normal stress being positive, brittle fracture is the mode of failure, and it is controlled by the mean normal stress itself. There is a condition or dividing line in between the two extremes that separates the ductile region from the brittle region. This is the ductile-brittle transition at $\alpha = 1$. At this condition both the deviatoric and the dilatational stresses influence the mode of failure in a very specific manner. This behavior at $\alpha = 1$ is consistent with that of classical fracture mechanics in the respect that for failure due to an opening mode effect caused by a particular normal stress, the magnitude of the failure level is unaffected by the presence or absence of small increments in the other stress components, (21). Thus, the behavior of a possible isolated crack in a material is just at the dividing line between the ductile and brittle behaviors. In the present theory then, the brittle range has classical fracture-like behavior due to isolated, cracks at one limit (bordering on ductile behavior) while at the other extreme, very brittle fracture behavior is controlled by dilatational effects. In the overall range of brittle material behavior, the fracture restriction, (22), applies.

The brittle range of behavior does not imply that all modes of failure have the typically brittle characteristics, but rather that as α becomes larger in the brittle range, the tensile type fracture modes of failure become more dominant and pervasive over a broader range of stress conditions. That is, the fracture criteria (22) and (23) apply over an increasingly broad range of stress conditions as α increases.

Evaluation

The Coulomb-Mohr form and the present criterion each possess general characteristics of both ductile and brittle behaviors and in that sense both are quite general. In the limit of the ductile range, the Coulomb-Mohr form becomes the Tresca criterion, while the present one becomes the Mises criterion. In the brittle range, both are capable of showing specific features of fracture. It is thus necessary to compare with established data sets in order to provide a decisive assessment of either of these criteria or of any other criterion.

The present general failure criterion (6)-(8) and (22) in complete component form is given by

$$\left(\frac{1}{\sigma_{11}^{T}}-\frac{1}{\left|\sigma_{11}^{C}\right|}\right)\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)$$

$$+\frac{1}{\sigma_{11}^{T}\left|\sigma_{11}^{C}\right|}\left\{\frac{1}{2}\left[\left(\sigma_{11}-\sigma_{22}\right)^{2}+\left(\sigma_{22}-\sigma_{33}\right)^{2}+\left(\sigma_{33}-\sigma_{11}\right)^{2}\right]\right.$$

$$\left.+3\left(\sigma_{12}^{2}+\sigma_{23}^{2}+\sigma_{31}^{2}\right)\right\}\leq 1 \tag{24}$$

and

$$\sigma_{1}\leq\sigma_{11}^{T} \quad \text{if} \quad \frac{\sigma_{11}^{T}}{\left|\sigma_{11}^{C}\right|}<\frac{1}{2} \tag{25}$$

where σ_1 is the largest principal stress for the stress state. The specific examples to follow are evaluated directly from (24) and (25).

First consider an example in the ductile range of behavior. Pae (1977) has produced general yield properties data for the polypropylene type of material (polymer). In addition to standard testing at atmospheric pressure, tests were conducted at states of elevated pressure. Table 1 shows the atmospheric pressure testing results. Yield properties σ_{11}^{T} and σ_{11}^{C} were used to calibrate the present yield/failure theory form (24-25) and the Coulomb-Mohr theory, as determined by the forms given by Paul (1968). The value of parameter α is 0.22 and the fracture criterion (25) is not found to enter this

data case. The maximum normal stress and maximum normal strain criteria are also given for historical interest. The former takes the form of a cube in principal stress space and the latter as a cube in principal strain space, both are offset with respect to the origins. Both criteria are calibrated by σ_{11}^{T} and σ_{11}^{C}. Poisson's ratio of 1/3 was taken in the case of the maximum normal strain criterion. Finally, the one parameter Mises criterion was calibrated by σ_{11}^{T} as would be standard practice for it. The theories were then used to generate the predictions shown in Table 1.

Next the calibrated theories were used to predict the yield behavior of the material at a level of 1 kbar superimposed pressure, which is over twice as large as any of the yield stresses in Table 1. The theoretical predictions for the various theories for the additional stress

$$\hat{\sigma}_{ij} \quad \text{— additional stress}$$

beyond that of the superimposed pressure are compared with the testing data in Table 2. Taking the measured values as base line, the error in the worst of the three predicted yield stress components, $\hat{\sigma}_{ij}$, in each case in Table 2 can be found and are shown below. A similar examination of the total stresses, σ_{ij}, including the pressure, also gives the corresponding errors as

	$\hat{\sigma}_{ij}$ Error	σ_{ij} Error
Present	7%	7%
Coulomb-Mohr	24%	24%
Max Normal Stress	48%	45%
Max Normal Strain	42%	57%
Mises	50%	63%

From this and Table 1, it is seen that the present criterion gives the best comparison with the data. It might be expected that at larger pressures, which probe further into the equation of state range, the present predictions would be less accurate and more complex, empirical forms would be needed.

Polypropylene is considered to be a ductile material and it certainly is one of the most ductile of all polymers. Its tensile and compressive yield strengths differ by only about 20%. Yet the results of Tables 1 and 2 show that to apply the Mises criterion to model these generally ductile materials would be in serious error. And, to use the two parameter maximum normal stress or maximum normal strain criteria would be virtually no better than the one parameter Mises criterion. Concerning the Coulomb-Mohr criterion, it does in the limit become the Tresca criterion which is considered to be at least reasonably satisfactory for ductile metals. Yet in this data example, still well within the ductile range of behavior but not near the ductile limit condition, the Coulomb-Mohr prediction of behavior is inadequate.

The data examples just given show the consistency of prediction for the present theory in the ductile range of behavior. There was no reason to re-examine the specific case of very ductile metals. It is well understood that the Mises criterion is the fundamentally correct form. It is now important to turn to brittle behavior examples for further evaluation.

Now consider the generally brittle behavior of cast iron. The specific data is from Cornet and Grassi (1955) on "inoculated" cast iron. The uniaxial tensile and compressive strength values are given by

$$\sigma_{11}{}^{T} = 56{,}000 \text{ psi}$$

$$= 386 \text{ MPa}$$

and

$$\sigma_{11}{}^{C} = -121{,}000 \text{ psi}$$

$$= -834 \text{ MPa}$$

The corresponding value of α is $\alpha = 1.16$ which while in the brittle range, (20), is not extremely so. This type of cast iron is "treated" to somewhat mitigate its brittleness. For biaxial stress states, the present prediction from (24) and the fracture cutoff (25) is shown in Fig. 4. The fracture cutoff zone from (25) is quite small and hardly discernable on the scale of Fig. 4. For any material example, as the value of α approaches $\alpha = 1$ from above, the fracture cutoff diminishes in size to the point of vanishing. Paul (1968) also gave a reasonable fit to the data in Fig. 2 using a three parameter model. The two parameter Coulomb-Mohr prediction for the case of Fig. 3 gives the corner of a "box" form in the first quadrant and a straight line connecting the intercepts in the fourth quadrant. It is not a good comparison with the data.

Next consider a very different material type which is a little further into the brittle range of behavior. The data of Ely (1965) is for rocket nozzle grade graphite processed by extruding granular forms of the material. The data were obtained by testing hollow cylinders in biaxial stress states, Fig. 5. The uniaxial tensile and compressive failure stresses are

$$\sigma_{11}{}^{T} = 3{,}800 \text{ psi}$$

$$= 26.2 \text{ MPa}$$

and

$$\sigma_{11}{}^{C} = -10{,}000 \text{ psi}$$

$$= -68.9 \text{ MPa}$$

The corresponding value of α is 1.63. The predictions from (24) and (25) are shown in Fig. 4. The Coulomb-Mohr prediction is about the same as that described in the preceding cast iron example, and it is not satisfactory. The fracture cutoff zone is more apparent in Fig. 5 than it was in the previous example.

The similarity of the failure forms of the greatly dissimilar materials, cast iron and graphite in Figs. 4 and 5, is here predicted by the particular and single property, α, with the reasonably close values of $\alpha = 1.16$ and 1.63 respectively. The closeness of these values is best seen on a logarithmic scale having the range from $-\infty$ to ∞ and with the two values being Ln $\alpha = 0.15$ and 0.49 respectively. The first example for polypropylene has Ln $\alpha = -1.50$. A geological material example will be given shortly with a Ln $\alpha = 2.63$.

Other examples can be posed which show very large differences between the Coulomb-Mohr method and the present method, but these generally have not been accessed by experimental investigations. In fact, the experimental difficulties are formidable, as will be further seen in the next evaluation example.

Geological materials in the upper layer of the earths crust, commonly rocks, exist generally in a state containing a large population of micro-cracks and they show all the characteristics of extreme brittle behavior. Brace (1964) has given data for predominately compressive states of stress that still have a portion of tensile behavior. Such a state is shown in Fig. 5 for normal stresses σ_{11} and $\sigma_{22} = \sigma_{33}$, in two quadrants, obtained by testing in a pressure chamber with superimposed effects. The rock type was Blair dolomite, with uniaxial tensile and compressive failure stresses as

$$\sigma_{11}{}^{T} = 0.34 \text{ kbar}$$

and

$$\sigma_{11}{}^{C} = -5.07 \text{ kbar}$$

The corresponding value of α is 13.9. The present prediction from criteria (24) and (25) is shown in Fig. 6. The upper portion is the fracture cutoff effect directly from (25). It is a sharply pronounced cutoff effect in contrast with the preceding two examples which were not nearly so deeply into the brittle range as is this example. Also shown is the Coulomb-Mohr prediction and it is not satisfactory. Rock mechanics fracture type forms with three or more parameters can generally be quite good in fitting rock failure data. The two parameter Coulomb-Mohr form is more problematic. All of the data in this field shown the fracture cutoff characteristic and a generally concave inward type of failure surface, although within these general features there can be extremely wide variations of results, Jaeger and Cook (1979). The two parameter Coulomb-Mohr form is not consistent with these general features but the present two parameter form is compatible. A complication that must be kept in mind when comparing with rock failure data is that there are serious questions as to whether geological materials can be modeled as being homogeneous and isotropic. Also, major uncertainty caused by the variability for tensile testing results must be emphasized. This same testing technique could be usefully applied to ceramics where the variability problems would be less severe.

Regarding the fracture dominated brittle range of behavior, there are very few comprehensive two parameter failure forms that show brittle behavior characteristics but still span the range to also include ductile behavior. Within this narrow grouping the differences between the present approach and the Coulomb-Mohr theory are striking. For example, the latter theory is independent of the intermediate principal stress, while the present theory shows a strong dependence on it in the brittle range. Of these two basic

two parameter theories, only the present one appears to provide a realistic correlation with testing results, not only in the brittle range, but also in the ductile range.

Conclusions

The broad scope of the failure characterization that is given here is made possible by the treatment that unifies yielding in the ductile range with fracture in the brittle range. The corners that appear on the yield/failure surface diagrams are due to the intersection of fracture modes and yielding modes of failure. The corners do not occur in the ductile range of behavior but are in the brittle range where the ductile and brittle modes compete for dominance. The brittle modes of failure are of two types, one being of the classical fracture mechanics form involving crack opening caused by a positive transverse stress component, and one related to the controlling effect of positive mean normal stress.

In the specific direction that is developed and followed here, the elastic energy form (5) and the yield failure criterion (6) are determined as dual relations originating from the same mathematical source. Together with the fracture restriction, (22), they form a self contained, reasonably general account of homogeneous isotropic material behavior. These three relations (5), (6), and (22) might be characterized as the elastic-inelastic constitutive relations within the scope of this theory. These relations must be supplemented by the usual other field variable relations. It is useful to briefly consider the four governing constitutive properties, E, ν, κ and α.

Properties E and κ have the same units as stress while ν and α are non-dimensional. All four properties are determinable by only uniaxial stress state testing. It is reasoned here that κ and α are properly designated as properties since they have a role far stronger than just that as curve fitting parameters. One could always assemble failure forms with more parameters than the two properties involved here. All existing failure forms with three or more parameters appear to be of empirical origin. The empirical forms would likely fit some sets of materials data better than the present two property theory, however, there would be no assurance that such empirical forms would fit all sets of data better for the many different classes of homogeneous materials.

Recognizing the occurrence of the ductile-brittle transition at $\alpha = 1$, (19), it follows that for the ductile range, $\alpha < 1$, (18), α and κ could be loosely designed as yield properties and in the complementary brittle range, $\alpha > 1$, (20), α and κ would be failure properties under many or most conditions. In the latter range, the term fracture properties could be used in place of failure properties with coequal significance. Of course, these designations are necessarily somewhat broad and the detailed designation of a failure mode type would depend not only on the state of the material, but also upon the exact stress state to which it is subjected.

We now return and conclude with the historical perspective with which this article began. There has been a two hundred year flow of contributions into three dimensional yield and failure theory. However, none of these formulations achieved comprehensive applicability. The present work, as well as work and efforts by many others, is generated by this timeless need to better understand the nonlinear response of materials, especially the consummate nonlinearity, physical failure.

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Table 1

Polypropylene Yield Strengths at Atmospheric Pressure

(Units of psi x 10^3 = MPa x 6.895)

	σ_{11}^T	σ_{11}^C	σ_{12}^Y
Data (Pae, 1977)	5.4	-6.6	3.5
Present			3.45
Coulomb-Mohr			3.0
Max Normal Stress			5.4
Max Normal Strain			4.1
Mises		-5.4	3.1

Table 2

Polypropylene Yield Strengths at Atmospheric Pressure + 1kBar

(Units of psi x 10^3 = MPa x 6.895)

	$\hat{\sigma}_{11}^{T}$	$\hat{\sigma}_{11}^{C}$	$\hat{\sigma}_{12}^{Y}$
Data (Pae, 1977)	8.9	-10.8	5.8
Present	8.8	-10.0	5.4
Coulomb-Mohr	8.0	-9.8	4.4
Max Normal Stress	6.4	-5.6	5.6
Max Normal Strain	5.7	-6.3	4.3
Mises	5.4	-5.4	3.1

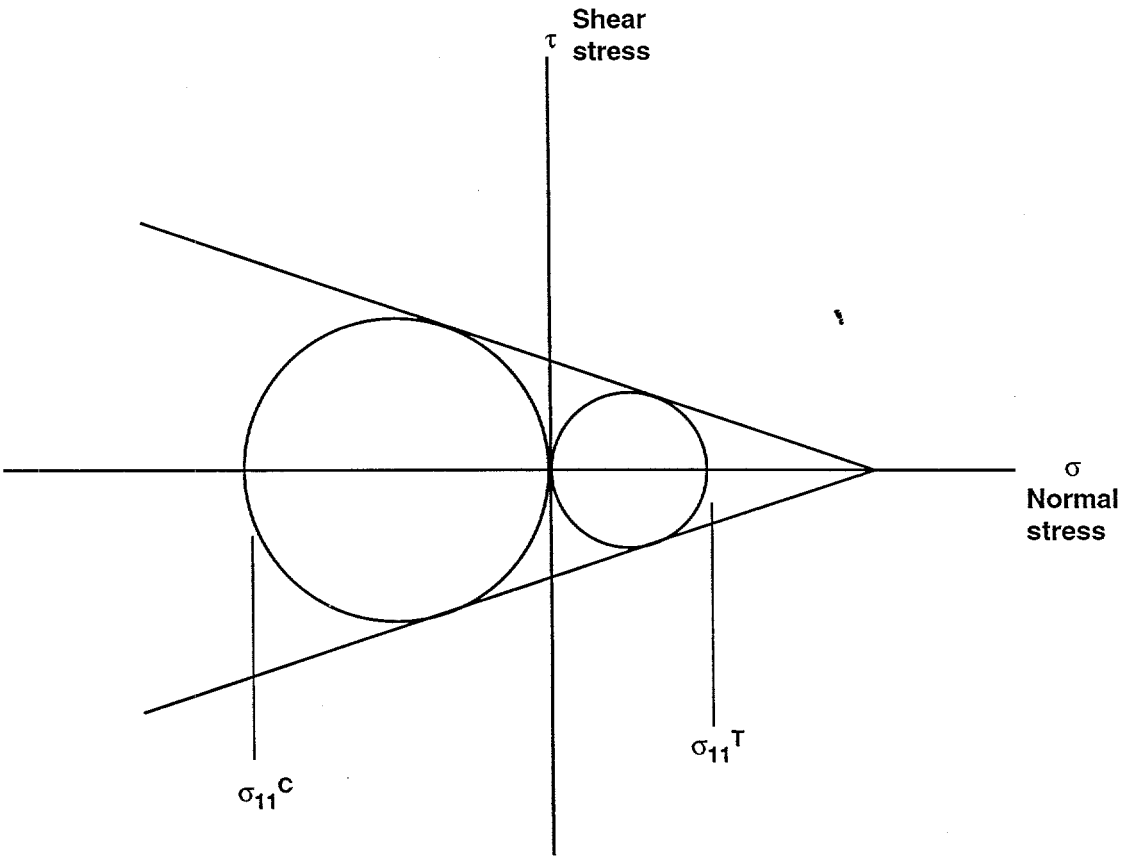

Figure 1. Coulomb-Mohr Failure Plane Normal and Shear Stress Envelope

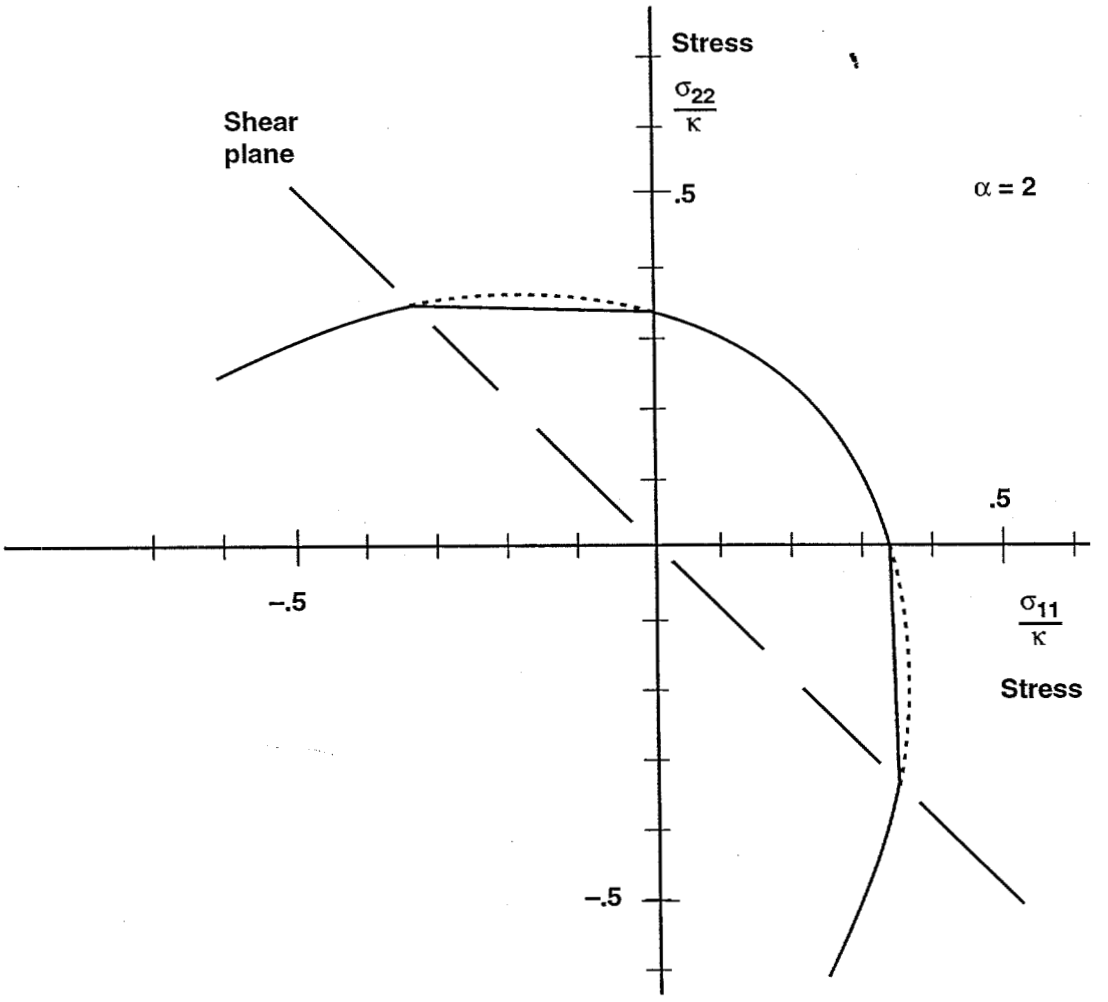

Figure 2. Fracture Cutoff Effect, Biaxial Stress State

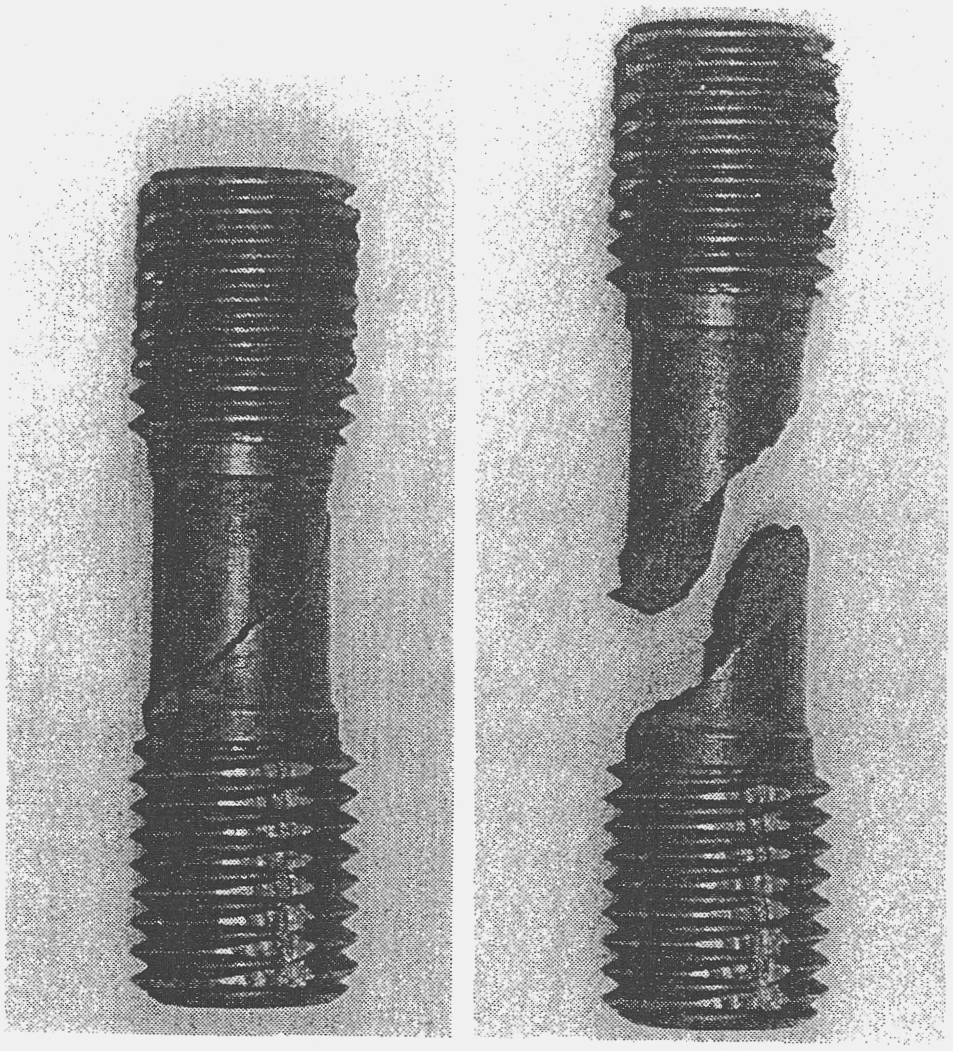

Figure 3. Torsion of a Cast Iron Speciman Showing 45° Tensile Fracture Surface Orientation, Popov (1990)

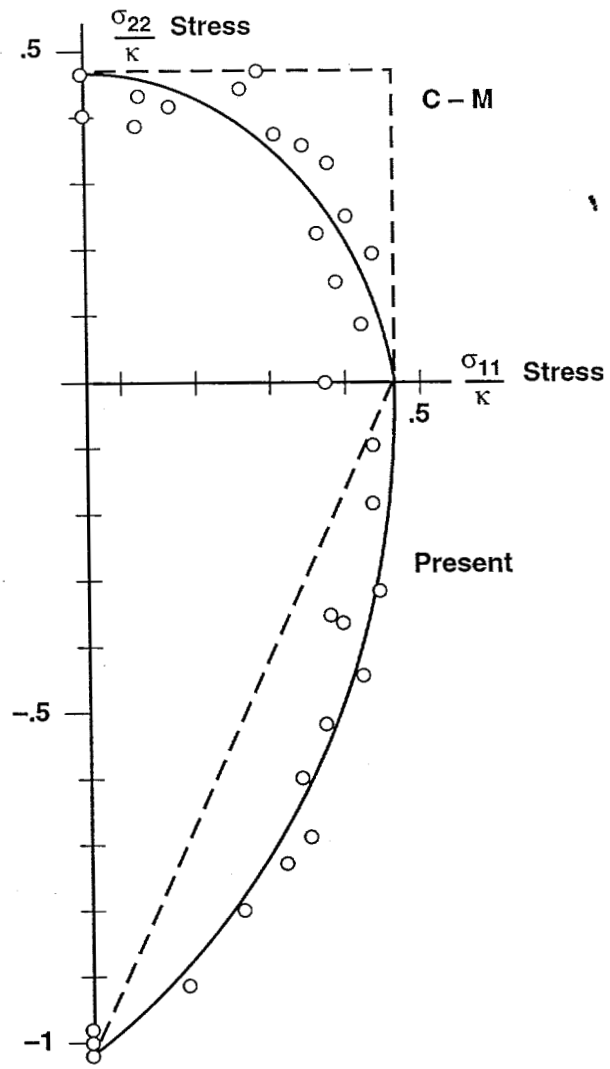

Figure 4. "Innoculated" Iron Yield/Failure, Cornet and Grassi (1955)

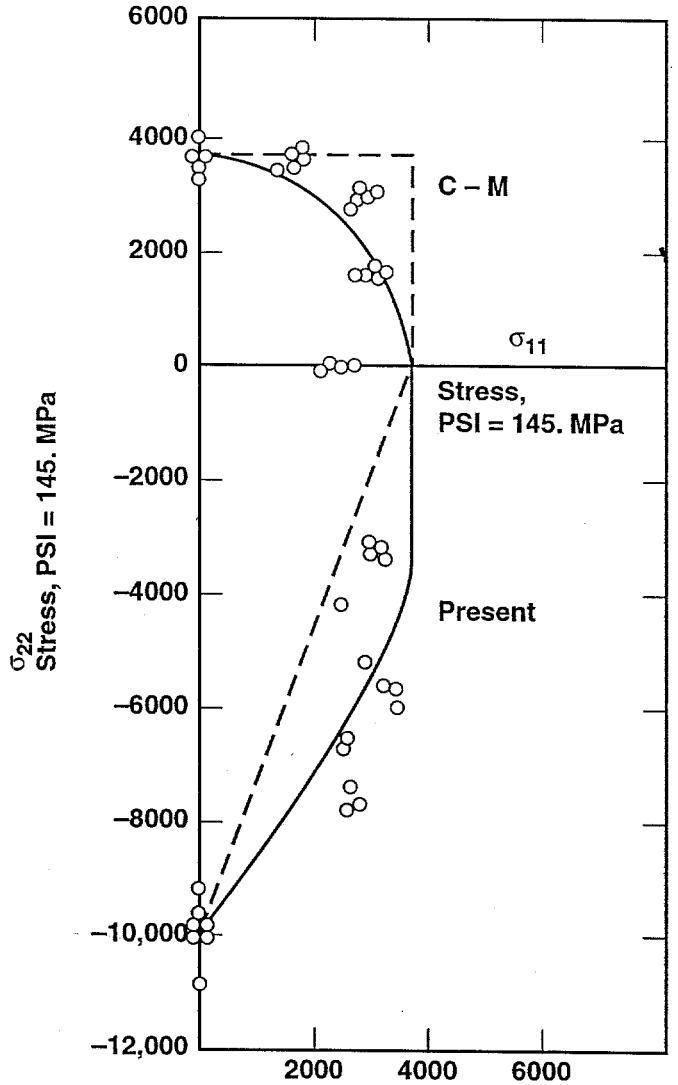

Figure 5. Graphite Yield/Failure, Ely (1965)

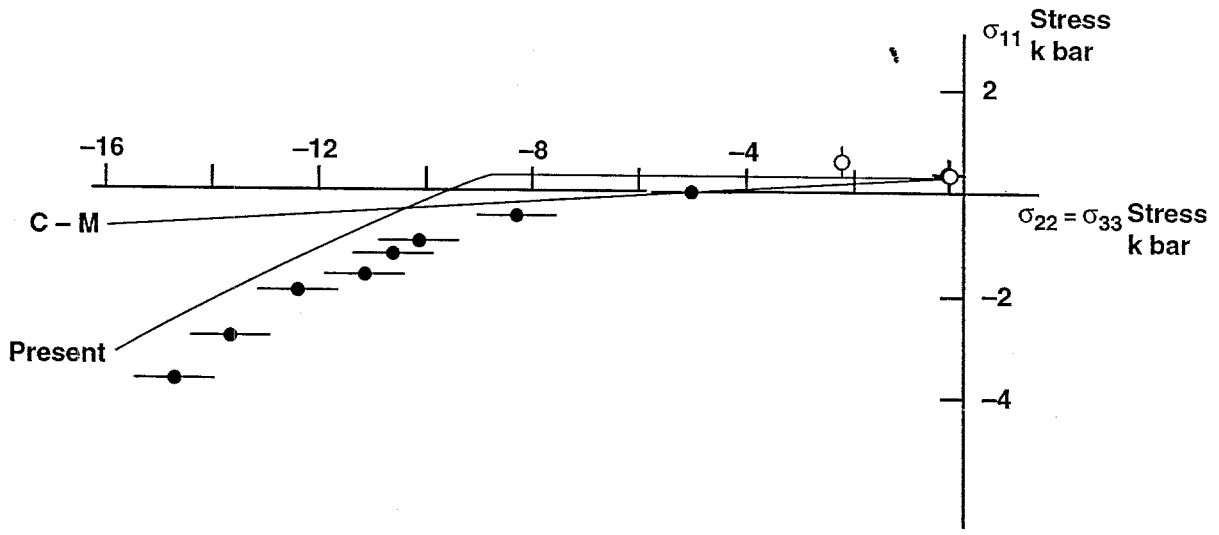

Figure 6. Blair Dolomite Yield/Failure, Brace (1964)