XXVI. THE PROOF OF THE SPLITTING MODE OF COMPRESSIVE BRITTLE FAILURE AND INTEGRATION WITH GENERAL FAILURE THEORY

1. Introduction

The subject here is that of materials failure, it has always been of utmost concern. The popular failure theory of the early days was disproved and laid to rest. All subsequent attempts have also proven to be not just inadequate but incorrect. Summaries of the many errant historical forays into failure theory have been given by Christensen [1].

Since the early times the problem of materials failure as a general theory has remained unfulfilled. Over the long term the situation has degenerated into a formidable obstacle to advancement. Only very recently has a new and viable failure theory for isotropic materials been set forth. It will provide the starting platform for the following work.

The overall purpose of this work is to endeavor to solve one of the most difficult and most important failure specific problems of all time. The problem concerns the uniaxial compressive failure for extremely brittle materials. That along with uniaxial tensile failure comprise the two most fundamental states of materials failure. Both are classical. The tensile problem is thoroughly and completely understood. The compressive problem, particularly in the brittle range of behavior, remains shrouded in mystery and uncertainty. The compressive failure problem will be fully posed in the next section after first recording the new failure theory and formalizing the brittle limit definition.

2. Problems of Materials Failure at the Brittle Limit

The main focus and attention is given here to the failure behavior of materials at and near the brittle limit. Failure problems are widely recognized to be among the most difficult problems in existence, and failure at the brittle limit must accordingly be regarded as exponentially difficult. But such problems are not impossibly difficult, as will be revealed here.

What is the meaning of the term “the brittle limit”? To begin to answer this question we first must go back to the meaning of the ductility of failure. Brittleness is the complete absence of ductility. Ductility is commonly taken to be expressed by the strain at failure for uniaxial tension. But that says absolutely nothing about ductility and brittleness in other stress states, such as uniaxial compression. This is not a productive line of investigation for the present purposes and some other direction must be found.

It is best to approach the problem somewhat indirectly. The approach followed by Christensen [1] was to search for an organizing principle that governs homogeneous, isotropic materials failure. Ultimately it was found to be the spectrum for the materials property ratio T/C where T and C are the tensile and compressive uniaxial strengths. The entire spectrum from the organizing principle is given by

$$0 \leq \frac{T}{C} \leq 1 \tag{1}$$

The ductile limit is obvious, it is specified by T/C=1.

Behavior at and near the ductile limit is very well understood. The T/C=1 case is governed by both the Mises and Tresca criteria. Originally the Tresca criterion was considered to be the fundamental form with the Mises case seen as an approximation to it. Over a period of about a hundred years this presumed understanding shifted and inverted to the completely opposite circumstance. Now days the Mises criterion is well understood to be the fundamental form at the ductile limit T/C=1. The Tresca criterion is merely an approximation to it.

With the ductile limit being very well understood, the organizing principle could only admit the diametrically opposite case as being the brittle limit, as specified by T/C=0. That is the operating principle at work here. Failure behavior near and at T/C=0 corresponds to failure approaching and at the brittle limit.

What are the implications of the materials type having T/C=0? This materials type or class must have T=0 and C≠0. Thus the material must be so damaged or so incoherent that it cannot support any uniaxial tensile stress. Does that mean that the material cannot support any stress state whatsoever? The answer is negative. Even though T=0 the fact that C≠0 asserts that it can sustain a uniaxial compressive stress up to a point before failing. Perceptive readers might see this as an inconsistency, a contradiction, but that perception would be incorrect.

If the brittle limit requires T=0 how can one rationalize the condition C≠0, at a finite value for the uniaxial compressive strength? The proposition goes like this: decompose uniaxial compression into its dilatational and distortional components, as follows

$$\begin{bmatrix} \sigma_{11}=-1 \\ \sigma_{22}=0 \\ \sigma_{33}=0 \end{bmatrix} = \begin{bmatrix} \sigma_{11}=-1/3 \\ \sigma_{22}=-1/3 \\ \sigma_{33}=-1/3 \end{bmatrix} + \begin{bmatrix} \sigma_{11}=-1/3 \\ \sigma_{22}=1/3 \\ \sigma_{33}=0 \end{bmatrix} + \begin{bmatrix} \sigma_{11}=-1/3 \\ \sigma_{22}=0 \\ \sigma_{33}=1/3 \end{bmatrix} \quad (2)$$

The last two groupings in (2) are two states of shear stresses. It is the first grouping on the RHS of (2) that is a compressive hydrostatic stress. It further is this compressive hydrostatic stress that imparts stability to the state of uniaxial compression even though the corresponding uniaxial tensile state and shear state cannot support any stress level.

At this point the stability argument is only a plausible conjecture. It must await the proof as being delivered by a full blown, verified field theory of failure.

It is not possible to go any further based only upon reasoning explicitly and exclusively related to the values of T/C from the organizing principle. The next step must be the assembly of a comprehensive and complete theory of isotropic materials failure. Subsequent to that, the first test of the failure theory must be to verify that it captures and allows the brittle limit of materials failure with T=0 and C≠0. Such a theory has been developed by Christensen [1]. It will be stated presently.

The status and importance of the brittle limit should be placed in its proper perspective. It is just as important as the ductile limit in terms of its significance for a general theory of failure. If any proposed general theory cannot recover the proper behavior at the brittle limit as well as at the ductile limit, then it really isn't a general theory. It's just another empirical exercise that possibly could be used for interpolation purposes with data but would be totally meaningless for projections beyond the range of the data.

The new theory of failure that will be employed here can now be briefly stated. The main failure criterion is the polynomial invariants form, also called the failure potential, as

$$\left(1-\frac{T}{C}\right)\hat{\sigma}_{ii}+\frac{3}{2}\hat{s}_{ij}\hat{s}_{ij}\leq\frac{T}{C} \tag{3}$$

where T and C are the uniaxial strengths and the stress is nondimensionalized by C as

$$\hat{\sigma}_{ij}=\frac{\sigma_{ij}}{C} \tag{4}$$

and where s_{ij} are the deviatoric stresses. The failure criterion (3) applies over the full range of T and C given by (1).

There also is an auxiliary (but important) fracture criterion that is competitive with (3). It applies over the partial range shown below

$$\begin{aligned}&\hat{\sigma}_1\leq\frac{T}{C}\\&\hat{\sigma}_2\leq\frac{T}{C}\quad for\quad 0\leq\frac{T}{C}\leq\frac{1}{2}\\&\hat{\sigma}_3\leq\frac{T}{C}\end{aligned} \tag{5}$$

and where $\sigma_1,\sigma_2,\sigma_3$ are the three principal stresses.

Amazing as it may seem, this comprehensive theory of isotropic materials failure is fully specified and made operational by only two independent failure properties, T and C. The proof of validity has been established in many separate papers and in the book [1]. The guides to these proofs of validity are given in Ref. [2], Section 3 of that reference. The derivation of failure theory (3)-(5) was quite involved and necessitated many different facets of technical considerations but the end result could not have turned out to be more simple. The brittle limit of (3)-(5) will be explained here and now.

In principal stress form and using dimensional stress then (3) and (5) become

$$\left(\frac{1}{T}-\frac{1}{C}\right)\left(\sigma_1+\sigma_2+\sigma_3\right)+\frac{1}{2TC}\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]\le 1 \qquad (6)$$

and

$$\begin{aligned} &\sigma_1 \le T \\ &\sigma_2 \le T \quad for \quad 0\le\frac{T}{C}\le\frac{1}{2} \\ &\sigma_3 \le T \end{aligned} \qquad (7)$$

The first task is to see if a rational and physically reasonable brittle limit evolves and can be extracted from this failure theory. From failure condition (6) at T/C=0 there remains

$$C\left(\sigma_1+\sigma_2+\sigma_3\right)+\frac{1}{2}\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]\le 0 \qquad (8)$$

Relation (8) is completely consistent at the brittle limit and the uniaxial compressive stress C remains to be determined independently from testing. At the brittle limit conditions (7) requires that all principal stress components must be negative, positive components cannot be supported. At the brittle limit the Mohr-Coulomb failure criterion becomes irrational.

The following section takes up the problem of interest here that can only be posed at the brittle limit. It determines the explicit failure mode in uniaxial compression at the brittle limit and also in its neighborhood.

Another classical problem also falls into the category of occurring solely at the brittle limit. It is the problem of the angle of repose. It will be taken up in future work.

For a material at the brittle limit, it is first necessary to establish that it absolutely can support a uniaxial compressive stress up to the point of a finite value of the compressive stress and in so doing determine the nature of the material failure process at the uniaxial stress equal to –C. To put this another way, very brittle materials in uniaxial tension suffer failure due to standard fracture. What exactly occurs with so called very brittle materials at their load limit in uniaxial compression?

3. Mathematical Proof of the Splitting Mode and Mechanism of Compressive Brittle Failure

The problem of prime importance and of direct interest is that of the elusive failure mode in uniaxial compression for materials at the brittle limit, T/C=0. Such materials are approached by typical geological materials having quite large values for C but very small values for T. These materials types were used to prove the inadequacy and incorrectness of the Mohr-Coulomb theory of materials failure.

Now we can approach the problem of determining the explicit mode of failure for the uniaxial compressive stress at the brittle limit. The first introductory part of this development comes from the work of Christensen [3] where for uniaxial tension and compression the associated angles of failure were determined. Write the form (6) in symbolic form as

$$f\left(\sigma_{ij}\right) \leq 1 \tag{9}$$

where f( ) is the failure potential. Take the strain increments of failure as given by

$$\dot{\varepsilon}_{ij}^{f} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{10}$$

and where λ is a dimensional property of the failure process in the material. Using (6) in (10) gives

$$\dot{\varepsilon}_{11}^{f} = \lambda \left(\frac{1}{T} - \frac{1}{C} + \frac{2}{TC} \sigma_{11}^{f} \right) \tag{11}$$

$$\dot{\varepsilon}_{22}^{f} = \dot{\varepsilon}_{33}^{f} = \lambda\left(\frac{1}{T} - \frac{1}{C} - \frac{1}{TC}\sigma_{11}^{f}\right) \tag{12}$$

Interest here is in the compressive case. Then (11) and (12) are given by

$$\dot{\varepsilon}_{11}^{f} = -\lambda\left(\frac{1}{T} + \frac{1}{C}\right) \tag{13}$$

$$\dot{\varepsilon}_{22}^{f} = \dot{\varepsilon}_{33}^{f} = -\lambda\left(\frac{2}{T} - \frac{1}{C}\right) \tag{14}$$

Now take the failure angle with the axial direction to be ϕ and take a rotated coordinate system as 1′ vs. 2′ with direction 1′ being in the plane of the failure and 2′ being normal to it as shown in Fig. 1.

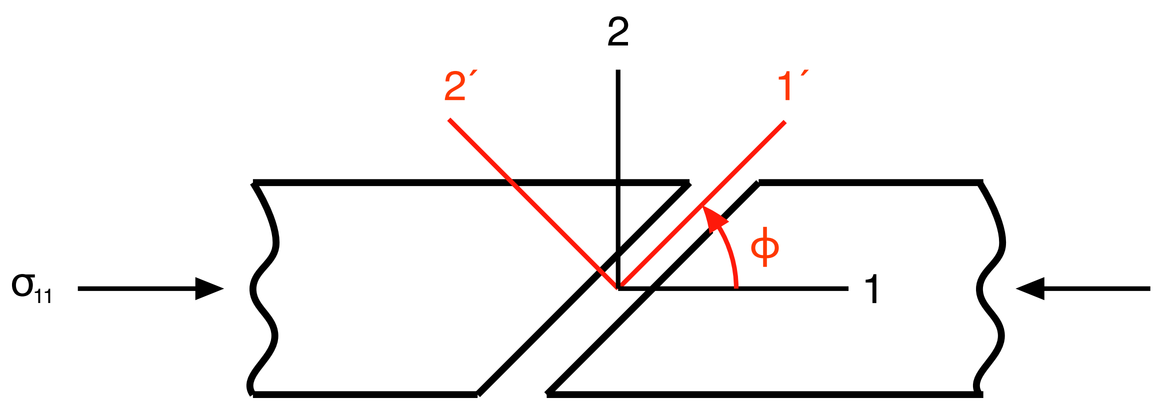

Fig. 1 The plane of failure in uniaxial stress

The strain increments in the rotated coordinate system are

$$\dot{\varepsilon}_{11}' = \lambda\left[-\left(\frac{1}{T} + \frac{1}{C}\right)\cos^2\phi + \left(\frac{2}{T} - \frac{1}{C}\right)\sin^2\phi\right] \tag{15}$$

$$\dot{\varepsilon}_{22}' = \lambda\left[-\left(\frac{1}{T} + \frac{1}{C}\right)\sin^2\phi + \left(\frac{2}{T} - \frac{1}{C}\right)\cos^2\phi\right] \tag{16}$$

$$\dot{\varepsilon}'_{12} = -\frac{3\lambda}{T}\sin\phi\cos\phi \tag{17}$$

During a shear band forming failure process the "in plane" failure strain increments $\dot{\varepsilon}'_{11}$ are taken to vanish while the "out of plane" failure strain increments increase or decrease in an unlimited manner. This gives the failure angle from (15), $\dot{\varepsilon}'_{11} = 0$, as

$$\tan^2\phi = \frac{1+\frac{T}{C}}{2-\frac{T}{C}} \tag{18}$$

From Ref. [3] the angles of the failure plane are given as in Fig. 2.

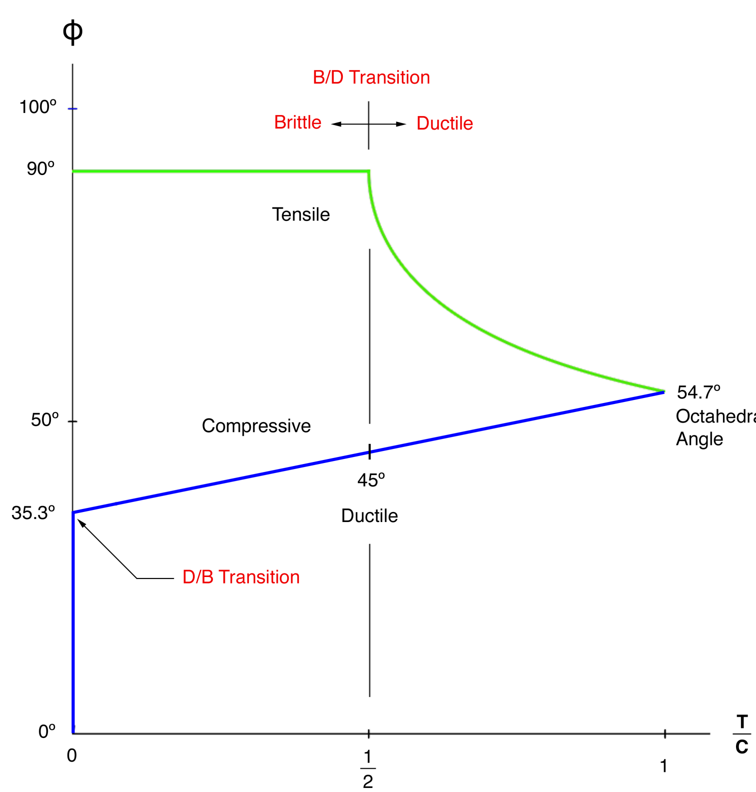

Fig. 2 Failure angles in uniaxial stress including only the shear band mode of compressive failure

The uniaxial tensile case is also shown in Fig. 2. The ductile/brittle transitions are as shown in Fig. 2. Please see Ref. [3] for a full and complete discussion on these matters.

The specific result of interest here is the compressive failure angle form shown in Fig. 2. Although it appears to be linear in form it is not, it is the form that results from (18). This mode of compressive failure up to the ductile/brittle transition shown in Fig. 2 is completely that of the formation of a shear band. Even at T/C=0 the angle of the shear band is at 35.3º from the axial direction and it involves $\dot{\varepsilon}'_{11}=0$, that is, there is no nonlinear normal strain increment in the plane of the shear band.

By supposition the splitting mode of compressive brittle failure occurs at the brittle limit T/C=0 and it involves the failure angle aligned with the axial direction $\phi=0$ accompanied by a lateral expansion in the failure process. The previous solution described above and involving the shear band mechanism of failure does not fit the prescription for a splitting mode of failure at the brittle limit. How can these two very different types of compressive failure behavior be reconciled? The answer and understanding to this fundamental question will now be developed.

In arriving at the result of ϕ=35.3º for the shear band mechanism of failure from (15) there was the condition $\dot{\varepsilon}'_{11}$=0 followed by letting T/C $\rightarrow$0. There is an alternative way to proceed. Before letting T/C $\rightarrow$0 take the limit process as having $\phi\rightarrow$0 followed by T/C $\rightarrow$0. From (15)-(17) the result from setting ϕ=0 is

$$\dot{\varepsilon}'_{11}=-\lambda\left(\frac{1}{T}+\frac{1}{C}\right) \tag{19}$$

$$\dot{\varepsilon}'_{22}=\lambda\left(\frac{2}{T}-\frac{1}{C}\right) \tag{20}$$

$$\dot{\varepsilon}'_{12}=0 \tag{21}$$

Note the similarity of these equations to (13) and (14). In arriving at (19)-(21) the condition $\dot{\varepsilon}'_{11}$=0 could not be used because that is only applicable to the shear band mode of failure.

Re-write (19) and (20) as

$$\dot{\varepsilon}'_{11} = -\frac{\lambda}{C}\left(\frac{C}{T}+1\right) \tag{22}$$

$$\dot{\varepsilon}'_{22} = \frac{\lambda}{C}\left(2\frac{C}{T}-1\right) \tag{23}$$

Finally, go to the brittle limit with T/C $\rightarrow$0. This gives (22), (23), and (21) as

$$\dot{\varepsilon}'_{11} \rightarrow -\infty \tag{24}$$

$$\dot{\varepsilon}'_{22} \rightarrow \infty \qquad for\, \phi \rightarrow 0,\ \frac{T}{C} \rightarrow 0 \tag{25}$$

$$\dot{\varepsilon}'_{12} = 0 \tag{26}$$

This grouping unequivocally is the splitting mode and mechanism of failure as occurring at T/C=0. The material undergoes crushing in the axial direction and suffers large lateral expansion in the explicit splitting mechanism and mode of failure.

This much takes care of what happens at T/C=0 but that still leaves open the question of what transpires over the entire range of T/C for the compressive mode and mechanism of failure as it relates to the splitting mechanism.

In examining the shear band mode of compressive failure it is found that $\dot{\varepsilon}'_{22}$ is of negative value over the range 1/2≤T/C≤1 but it goes to zero right at T/C=1/2. The expectation is that $\dot{\varepsilon}'_{22}$ must be positive over the range 0≤T/C≤1/2 to be consistent with the splitting mechanism of failure. We will operate with that presumption, subject to later verification. With that condition it follows that a splitting mode type of failure occurs from $0 \le T/C \le 1/2$. In following this direction to proceed it also follows that for the splitting mode of failure over the range $0 \le T/C \le 1/2$ the condition $\dot{\varepsilon}'_{11} = 0$ cannot be used.

For the relationship of ϕ to T/C over the splitting mode range of behavior, the controlling form will be taken to have the same general form as occurs for all the other modes of failure occurring in Fig. 2. This form is that of

$$\tan^2\phi = \frac{a + b\frac{T}{C}}{c + d\frac{T}{C}} \tag{27}$$

where a, b, c, and d are to be determined, but only three of them are independent.

The form in (27) becomes much simpler when the necessary condition (24)-(26) that

$$\tan^2\phi \to 0 \quad as \quad \frac{T}{C} \to 0 \tag{28}$$

is invoked. Using (28) in (27) then gives the required solution as having the form

$$\tan^2\phi = \frac{\frac{T}{C}}{c + d\frac{T}{C}} \tag{29}$$

Determine c and d in (19) to give continuity of $\tan^2\phi$ from (18) and from (29) and also require continuity of the associated first derivatives. The end result is the mode of compressive failure given by

$$\tan^2\phi = \frac{3\frac{T}{C}}{1 + \frac{T}{C}} \quad for \quad 0 \le \frac{T}{C} \le \frac{1}{2} \tag{30}$$

It can be verified that $\dot{\varepsilon}'_{22} \geq 0$ for the splitting/shearing mode of (30).

The full form of the failure angles including the splitting/shearing mode (30) is shown in Fig. 3.

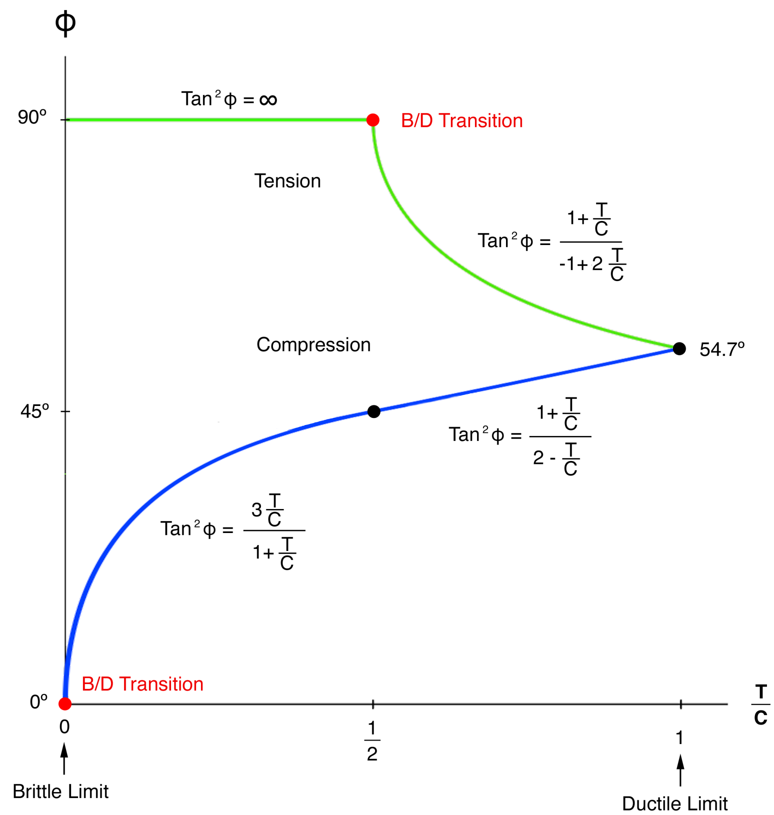

Fig. 3 Failure angles in uniaxial stress including the splitting mode of compressive failure

In Fig. 3 the shear band mode of compressive failure controls the range from $1/2 \leq T/C \leq 1$ whereas the combined splitting/shearing mode of compressive failure controls the range $0 \leq T/C \leq 1/2$.

Fig. 2 shows the shear band mode of compressive failure behavior whereas Fig. 3 shows the somewhat more involved form of compressive failure in which the combined shear band and splitting mode of failure participates. Both separate cases of Figs. 2 and 3 are legitimate representations of possible failure types. However, the form in Fig. 2 is superseded by that in Fig. 3 as the more common form since it fully admits the splitting mode of failure. Fig. 2 is more restrictive.

4. Consequences for General Conditions of Failure

Fig. 3 encapsulates and conveys the full and complete range of failure angle behaviors for isotropic materials in the most basic of all stress states, uniaxial tension and compression. It is the purest of all extracts from the failure theory. It is unique in such matters. No other theory comes even close to such specificity, completeness, and rigor for a general treatment of materials failure.

At the limit T/C=1 the tensile and compressive failure angles are identical at the octahedral angle in Fig. 3. At the other limit, T/C=0, the two failure angle types have diverged to the maximum possible separation, one at 90º and the other at 0º. Most importantly, the location and significance of the two ductile/brittle transitions (or brittle/ductile transitions) have been brought to full and consistent account in Fig. 3.

Even though the failure mode in Fig. 3 from T/C=0 to T/C=1/2 is termed as the splitting/shearing mode, it is still considered to be ductile and only changes to the brittle behavior at the brittle limit T/C=0 where the ductile/brittle transition exists, see Ref. [3]. This mode of failure is a combination of the shear band behavior and the splitting behavior. The pure splitting mode of failure only occurs at the limit of ϕ=0 and T/C=0 for the compressive failure.

Specific examples from the theoretical splitting angle result (30) are as follows

$$\begin{aligned} &At\,\frac{T}{C}=\frac{1}{75}=0.0133 \quad \phi=11.2^{\circ} \\ &At\,\frac{T}{C}=\frac{1}{50}=0.02 \qquad \phi=13.6^{\circ} \\ &At\,\frac{T}{C}=\frac{1}{25}=0.04 \qquad \phi=18.8^{\circ} \end{aligned} \tag{31}$$

These are within the usual range for geologcal materials and indicate that the associated splitting angles are from about 10-20 degrees for such materials.

The splitting mode of compressive failure has always been much in evidence, especially with geological materials, Fairhurst and Cook [4]. A great many papers have been written on the splitting mode of failure. None have proceeded along the lines developed here. More on this will be said shortly. General aspects of rock type mineral materials are given by Jaeger, Cook, and Zimmerman [5].

The definitive experimental work on the splitting mode of failure was given by Chen and Ravichandran [6]. With a brittle glass ceramic material, they performed a series of elegantly designed experiments, some involving lateral confinement and some only involving free uniaxial compression. The splitting mechanism clearly emerged for the latter cases but not for the former ones. The material, of the trade name Macor, has a T/C value of about 1/10 placing it in the very brittle category. Most geological materials are even much more extreme.

Much of what was done here on the splitting problem has been related to the ductile/brittle transition. It was not directly involved in what was done but it was always of presence and influence. The two ductile/brittle transitions are highlighted in Fig. 3. It would be expected that the forms of these two transitions in Fig. 3 would exhibit a considerable degree of smoothing in the reality of testing evidence, nevertheless they are of first importance.

Anyone who has some familiarity with strength testing data would easily find agreement with the tensile ductile/brittle transition shown in Fig. 3 as occurring at T/C=1/2. It is the compressive ductile/brittle transition at T/C=0 in Fig. 3 that is especially fascinating and significant in the present context. There is a state of very low ductility in the near region of the ductile/brittle transition at T/C=0, Ref. [2]. This region of low ductility is exactly where the splitting mode of failure is the most active and prevalent. Away from this region, the state of uniaxial compressive failure is in the ductile to very ductile range, again please see Ref. [2].

Much or most of the past work on the splitting mode of brittle failure seeks to relate that effect to a fracture mechanics mechanism, even though the stress state is that of uniaxial compression. In sharp contrast, the present work characterizes the splitting mode of failure as the sudden and brittle (or very nearly brittle) collapse of the material with crushing in the axial direction and expansion in the lateral direction. It is the ductile/brittle transition at T/C=0 that ties it all together. The pure splitting mode of failure at T/C=0 is transformed or transitioned to exactly the brittle form of activation and near to T/C=0 it is almost brittle with very little ductility.

As mentioned, the failure behavior for brittle materials in uniaxial compression is sometimes loosely described as being that of fracture. In view of the present work that actually is a misleading view of the complex failure processes that actually occur and result in the splitting manifestation.

An intensive evaluation of the ductile/brittle transition has been conducted by Christensen, Li, and Gao [7]. That work successfully predicted the full implications of the overall behavior of ductile/brittle transitions in isotropic materials failure. Closely related to that work, an in-depth examination of the shear bands and the voids nucleation modes of initiating failure has also very recently been given by Christensen, Li, and Gao [8]. No doubt it is the local voids nucleation mechanism that leads to the splitting failure mechanism at a larger scale. All these matters show the important and even crucial role played by the ductile/brittle transitions in materials science and in materials failure in particular.

We close this paper with some perspective on this problem and its solution, especially as it relates to the times of antiquity. For the colossal temples of ancient Greece (and before) the massive supporting marble columns could possibly have failed by the compressive splitting mechanism analyzed here or by Euler buckling. Those two failure mechanisms, even if only understood heuristically at the time, were sufficient to allow the gifted designers and constructors to proceed. After all these centuries it is satisfying and helpful to better understand the explicit and complete failure mechanisms. There no longer is any reason or excuse for a heuristic, non-scientific approach to treating materials failure.

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