XIII. PROBABILISTIC FAILURE AND PROBABILISTIC LIFE PREDICTION

This section involves the application of probabilistic methods to the prediction of failure for materials. Only the most severe and demanding of these types of problems are considered, namely those of creep rupture and fatigue. Most other types of failure problems can be treated without the complications of statistical specification. Before explicitly getting into these "extreme" problems, some vital preliminaries must be developed. These concern not only the advantage but the necessity of using power law forms to achieve generality in some situations, and also concern the most important probability distribution of all for materials applications, the famous Weibull distribution. A brief introduction poses the problems of interest.

Introduction

All physical behaviors have variability and this is especially true with materials. The amazing and extremely fortunate thing is how well deterministic theories capture most of what is going on. Still there are some situations where the behaviors are so non-predictable by deterministic methods that statistical inference must be employed. Quantum mechanics is a prime example.

Usually when one averages over small scales to get behaviors at large scales the variability features blend into an effective continuum of average behaviors. This is the case with most macroscopic behaviors. In particular, for the failure of high quality materials, the static strengths are found to be tightly grouped around mean values that almost always suffice for predictions in applications. But there are exceptions, even at the macroscopic scale.

Failure conditions and specifications involve more than just the time independent quasi-static strength. Time can be involved in explicitly ways that are seemingly difficult to describe and understand at a simple level. Historically it was commonly observed that glass under low load levels

could creep and ultimately even rupture. The lifetimes could be years or even centuries. The slow flow of glass is called creep and the ensuing failure is called creep rupture. In modern terms many materials that are normally thought to be completely inert and subject only to static failure, actually undergo creep and creep rupture. All polymers fall under this category, as do metals at high temperature. The fascinating thing about this condition is that creep rupture in such materials can occur with little or no observable macroscopic creep preceding it. Obviously processes of degradation are moving along at scales below those of macroscopic observation.

Completely similar situations occur with materials subjected to cyclic loading, this is the broad and important area of the fatigue behavior of materials. Virtually all materials types are susceptible to fatigue failure. So there is far more to failure than just static failure.

When one gets into these broader categories of failure, one finds that the common treatments and conditions that apply with static failure do not apply with these more general failure types. In particular the tight grouping of failure data within a narrow band suddenly no longer seems to operate. The rule with these generalized failure types is that the test data are very widely scattered. The mean value of time to failure is almost meaningless, or at least it misses the main aspect of behavior, the spread. It is one of the ironies of technical history that fatigue has nearly always been and usually still is represented as single curve in an S-N plot that implies that mean value based on a few data points of fatigue life are perfectly acceptable. The reality is that typical fatigue data has so much scatter that it is obviously in need of a fully probabilistic treatment.

At this point the question then arises as to when can failure data be treated using only mean values and when must a full blown probabilistic treatment be employed. There is no simple and direct answer to this question but the following can be said. Whenever logarithmic scales are needed to display meaningful data sets, then that is a quite clear indicator that it may be necessary to account for extreme variability of the material response. In particular, when involved with matters of creep rupture lifetimes and fatigue failure lifetimes a probabilistic approach is certainly called for.

This section will examine failure problems that require the use of probability theory. Mainly these will be concerned with creep rupture and fatigue. Actually, it will be shown that both of these problem areas and methods of solution devolve from the same formalism. One is measured explicitly in terms of elapsed time while the other is in number of cycles, which is an equivalent scale for considering the accumulation of damage ultimately culminating in failure.

Although many different probability functions have been applied to these classes of problems, the attention here will be focused upon only one of these, the Weibull distribution. From a mathematical point of view the Weibull distribution is just one of a great many different probability distribution functions, but from a materials point of view it has a very special status. Its special meaning and applicability will be full explored here. The probability function of the Weibull type will be derived here in a manner that naturally displays its significance. Thereafter it will be applied to the life prediction problems of interest.

The entry to the lifetime prediction problem is given by the time controlled growth of flaws and defects. A theoretical approach appropriately named kinetic crack growth theory will be developed and applied to the lifetime problem, first in deterministic form, then with generalization to probabilistic forms.

Before embarking on this life prediction course an even more basic concept must be developed and assimilated. This concept is that of the power law form. Power law forms are widely employed in physics as representations for various effects linking descriptive variables. Power laws for specific applications to materials problems will be given close examination and development. They are not merely empirical forms, attractive only for their ease of use. In the present materials context they will be found to be indispensible. It is thus necessary to start with the development and basis for the power law form in materials failure applications, even before deriving the Weibull distribution.

Power Law Failure Interpretation

Power law behavior is commonly seen with the non-destructive properties of polymers, namely relaxation functions and creep functions. This account is begun by seeking to understand the source for the power law behavior with the failure of polymers.

When dealing with polymeric materials behavior, the relaxation functions and creep compliance functions are often represented through their spectra, Ferry [1]. It is equally well motivated to represent creep rupture (failure) behavior through spectra using the same formalism, thus take

$$\sigma(t_c) = \int_0^\infty \tilde{H}(\tau) e^{-\frac{t_c}{\tau}} d\tau \qquad (1)$$

where the constant stress causing failure at time t_c is σ and $\tilde{H}(\tau)$ is the spectrum.

For a single mechanism of failure there would be a single exponential describing the failure time form,

$$\sigma(t_c) = A e^{-\frac{t_c}{\tau_1}} \qquad (2)$$

The spectrum is then that of a single delta function at $\tau = \tau_1$. The simple form (2) functions very poorly in representing actual creep rupture behavior. This in itself gives a lead on how to search for a more general form.

Before proceeding further, it is helpful to recognize the advantage of using log scales and following Ferry [1] take (1) in the alternate form

$$\sigma(t_c) = \int_0^\infty \hat{H}(\tau) e^{-\frac{t_c}{\tau}} d(\log \tau) \qquad (3)$$

where

$$\hat{H}(\tau) = \tau \tilde{H}(\tau) \qquad (4)$$

and where the actual creep rupture failure process is now recognized to involve a continuum of changing failure modes, not just a single event.

The single delta function type of spectrum is singularly different from the function it attempts to represent, $\sigma(\tau_c)$. This single delta function is one limiting case and it works poorly. One could add more delta functions spaced at different increments of time but instead it is most helpful to go to the opposite limiting case by taking the spectrum and the function it represents as being identical, to within a scaling constant. This will give the most regular and continuously distributed spectrum.

Following this course take

$$\hat{H}(\tau) = \alpha \sigma(t_c)\big|_{t_c = \tau} \tag{5}$$

where α is some constant.

At this juncture there is a useful approximation to the spectra forms in (1) and (3). This is obtained by approximating the exponential in (1) by a step function that is positioned such that the total square error between the two is minimized, see Christensen [2]. Then the resulting form can be differentiated to obtain

$$\tilde{H}(\tau) \cong -\frac{d\sigma(\tau)}{d\tau} \tag{6}$$

with t_c replaced by τ. A time scaling factor of Ln $2 = 0.693$ is involved in this approximation but it is not needed here. This approximation is valid for large values of τ but not for small τ's. For the logarithmic scale case then

$$\hat{H}(\tau) \cong -\tau \frac{d\sigma(\tau)}{d\tau} \tag{7}$$

Next substitute (5) into (7) to obtain

$$\frac{d\sigma(\tau)}{d\tau} + \alpha \frac{\sigma(\tau)}{\tau} = 0 \qquad (8)$$

The solution of (8) is

$$\sigma(\tau) = B\tau^{-\alpha}$$

or

$$\sigma(t_c) = Bt_c^{-\alpha} \qquad (9)$$

Thus the power law form is the limiting case where the function and its spectrum are related by (5).

The power law form (9) has the best possibility to model physical behavior over many decades of time. The power law form is not just an empirical term used only for its convenience in manipulations. It has a physical basis as the smoothest, most regular descriptive form for distributed failure mechanisms activated by stress. Power law forms arise in a variety of situations involving materials failure. Ample use, even crucial use, will be made of power law forms in the following developments. In fact it will be found to provide the "spine" for the Weibull distribution.

Weibull Distribution Physical Basis

The Weibull distribution for use in probability theory is of surprisingly recent formalization and implementation, Weibull [3], 1951. This seminal paper is immensely readable, enjoyable, and convincing in its application to many different systems and situations. A general and complete treatment of the subject is now available and given by Rinne [4], including historical antecedents. The validity of the Weibull distribution for any of a wide variety of applications usually follows from the standard probabilistic approach: if it fits the data then it is used, if it doesn't then some other distribution function must be tried. It does work surprisingly well in a great many cases.

The explicit justification for the use of the Weibull distribution with materials failure problems usually appeals to the weakest link argument involving the weakest link (probabilistically) in a chain of links. It can give one pause to read in some books that this extremely simple one dimensional pattern of a chain of links provides the supporting foundation for the use of the Weibull distribution with the failure behavior of three dimensional continua, namely all homogeneous materials. Its applicability and success in the technical area is not in question, what is missing however is a substantial basis or development of why it is so successful. This seeming paradox will be addressed here to help solidify its use with materials. The investigation is at the macroscopic scale of most applications.

First, some appropriate terminology must be established. Some of the examples will use time as the independent variable. A collection of material samples under load of some type will failure in a sequential manner as time elapses. This could be the service life of electric lights or the function of turbine blades, or just anything that does not have absolutely perfect predictability. Whatever it is, serving in whatever environment, the cessation of function is defined as failure, and the items of interest are taken as a random variable of time. The times of failure can be gathered together, and the total number of failure up to any specific time are know. The collection of data suggests the existence of a Cumulative Distribution Function CDF (of failure) as shown in Fig. 1.
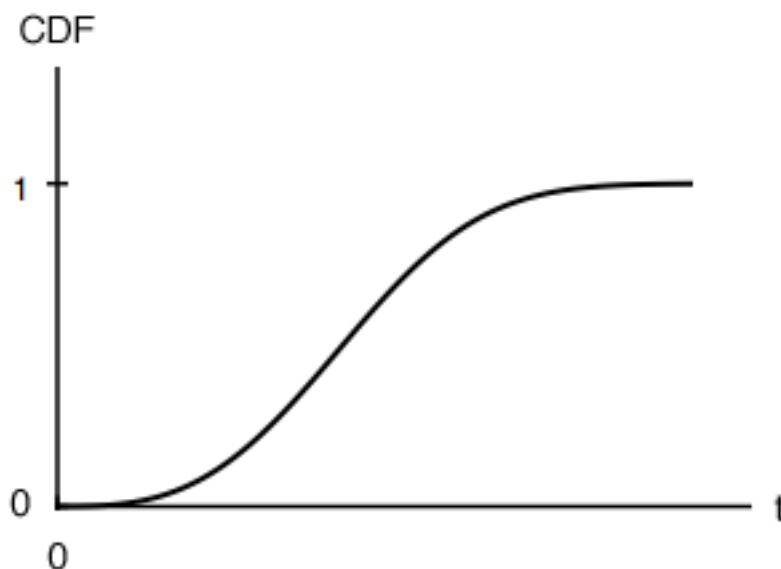
Fig. 1   Cumulative distribution function

The derivative of the CDF with respect to time forms the Probability Density Function, PDF. It shows the band of the dominant failure occurrences and often falls off on either side of its peak. The Complement of the Cumulative Distribution Function is given by (1 – CDF) and it provides the time record of the probability of survivors. Time could be replaced by any other variable of interest. Stress as well as time of course will enter all matters here.

It is tremendously advantageous to characterize these forms by analytical functions, the normal distribution being the most common PDF and CDF. Such distributions can then be use to predict a variety of effects. Predictability is the operative term. Even though the absolute time to failure cannot be determined with certainty, the probability of failure and of survival can be expressed quantitatively and used with complete assurance.

In addition to the PDF, CDF and 1 – CDF, there is a fourth basic function that will be of relevance here. From reliability theory the hazard rate or hazard function is defined by

$$h = \frac{\frac{d}{dt}(CDF)}{1 - CDF} \qquad (10)$$

The hazard function is defined as the instantaneous rate of failure since the denominator in (10) quantifies the survivors at the value of time t. To avoid confusion with the terminology of the instantaneous static strength, the hazard function will be referred to here simply as the rate of failure, rather than the instantaneous rate of failure. The term rate of failure applies whether it is the rate with respect to time or with respect of increase in stress, or anything else that is changing in a controlled manner.

With these terms, the derivation of a CDF for the failure of materials can begin. Rather than starting with the time dependent failure that occurs in creep rupture (or fatigue) it is helpful to start with the simplest case, that of the static failure of materials. The complications of time dependent creep

rupture will come later. Time is not explicitly involved in determining the static strength of materials since it is normally done quickly compared with the time scales of interest in creep rupture. Accordingly this will be called the instantaneous static strength or simply the instantaneous strength or even just the static strength, and it will be designated by σ_i. Normally this is taken to be a single scalar representing the mean value from a few tests of specimens. It is advisable here, however, to give full attention to the variability in the instantaneous strength because this may provide a clue or lead to the more serious effects that will follow in the time domain. The hazard rate, rate of failure, for the instantaneous static strength is then

$$h = \frac{\frac{d}{d\sigma}(CDF)}{1 - CDF} \tag{11}$$

The objective now is to determine the CDF for the instantaneous static strength of materials by any means that recognizes and respects the characteristics and capabilities of modern, high quality materials, whether they be metals, polymers, ceramics, glasses, or anything else.

Only the most regular forms are expected to be likely to apply to these engineering materials. This smoothness or regularity of behavior is consistent with that for many (but not all) physical systems. Furthermore, it is far easier and more direct to specify realistic forms for h, the rate of failure function, than it is to conjecture forms for the PDF or CDF. This is because the hazard rate or (probabilistic) rate of failure has a physical meaning that can be grasped intuitively, as will be seen.

There is no method to identify a unique form for the rate of failure, h, of the static strength but having just seen in the preceding sub-section the "power" of the power law representation, it is not only reasonable but compellingly logical to start with the power law form for h. Let h be given by

$$h = A\sigma^p \tag{12}$$

where p is the power law exponent. The allowable range for p for the instantaneous strength is not yet apparent, but it is seen that (12) embodies several different possibilities, as shown in Fig. 2.
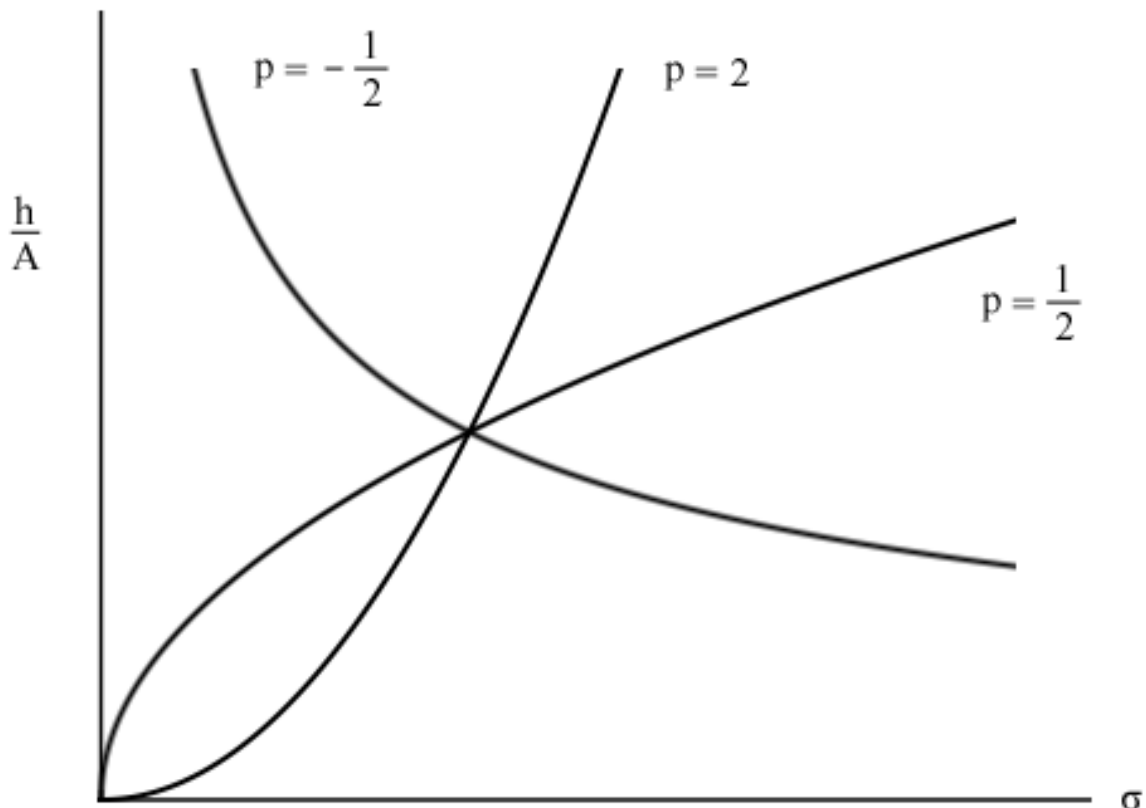

Fig. 2   Hazard (rate of failure) function for (12)

The three cases shown in Fig. 2 involve: (i) an increasing rate of failure with an accelerating rate of failure, (ii) an increasing rate of failure but with a decelerating rate of increase, and (iii) a decreasing rate of failure. All three distinctly different behaviors are controlled and specified by the single parameter, p. Relation (12) also admits special behaviors and interpretations at $p = 0$ and $p = 1$, as will be seen.

   The behavior shown in Fig. 2 for the power law form of h provides the physical basis for what will turn out to be the Weibull distribution. The probabilistic rate of failure (12) as a function of σ stress (and stress change)

is a monotonically changing function of σ with a monotonically changing first derivative. This is by far the most regular and most likely physical occurrence for the static strength behavior of engineering materials of widespread application. The Weibull distribution is not the only one with a monotone rate of failure form but it does appear to be by far the most versatile and most general two parameter form of all of them.

For a rate of failure that decrease with increasing stress surely the resulting distributions would be broad and diffuse. Conversely for a rate of failure that increases with increasing stress the distributions would seem to necessarily be narrow and concentrated. Exactly these physical effects will be found to occur with the CDF that results from the power law form for h, (12). Intuitively, it would be expected that most engineering materials that are highly enough developed to admit the characterization of being homogenous (uniform at the scale of intended use) would fall within category (i) above, having an increasing probabilistic rate of failure and with acceleration with respect to increasing stress. Interestingly, in contrast to the static strength, lifetimes will be found to conform to the decreasing rate of failure form as time increases.

Combining (11) and (12) gives

$$\frac{\frac{d}{d\sigma}(CDF)}{1 - CDF} = A\sigma^p \tag{13}$$

The solution of this differential equation is given by

$$CDF = 1 - e^{-\left(\frac{\sigma}{\sigma_s}\right)^{p+1}} \tag{14}$$

where

$$\sigma_s = \left(\frac{p+1}{A}\right)^{\frac{1}{p+1}} \tag{15}$$

Finally, let

$$p + 1 = m \tag{16}$$

which then gives (14) as

$$CDF = F(\sigma) = 1 - e^{-\left(\frac{\sigma}{\sigma_s}\right)^m} \tag{17}$$

where the power law form (12) from which this is derived has

$$A = \frac{m}{\sigma_s^{\,m}} \tag{18}$$

$$p = m - 1 \tag{19}$$

The CDF in (17) is that of the two parameter Weibull distribution with

$$m - shape\ parameter$$
$$\sigma_s - scale\ parameter$$

Large values for m give a very narrow, concentrated PDF while small values give very broad distributions. The values m = 1 (p = 0) is that for the exponential distribution while m = 2 (p = 1) gives the Rayleigh distribution. The shape parameter must have m > 0, so then the associated hazard rate power law form (12) has p > -1.

Thus the Weibull distribution is derived from the most fundamental form for the hazard, rate of failure function. The power law form (12) for the rate of failure has a very wide range of realistic physical behaviors and consequently and subsequently provides the very powerful formalism of the Weibull distribution. The chain of links scenario was of no use or relevance in this derivation.

Kinetic Crack Theory and Life Prediction

The long time behavior of materials in creep and fatigue conditions is very important for a wide range of applications. Unfortunately these are some of the most difficult problems that can be confronted insofar as the basic mechanisms of damage and failure are concerned. Nevertheless it is necessary to treat these problems, especially since there is so much scatter in typical testing data for these problems.

The ultimate goal is to obtain a full and complete probabilistic treatment for these problems but the initial approach will be to derive deterministic forms for the subject of life prediction, mainly aimed for creep rupture in polymers and metals at high temperatures, and for fatigue in all materials types. The approach uses the theory of kinetic crack growth, as small scale damage, and generally follows Christensen and Miyano [5], which follows much earlier work. The first part also has some overlap with the cumulative damage treatment in Section IV. The notation is changed slightly from that in Section IV to accommodate probabilistic variables.

Take an elastic material as having an initial state of flaws, here idealized as that of the central crack problem under Mode I conditions with the initial crack size of $2a_0$. Rapidly applied loads will cause instability of the crack due to fracture. The stress at which this occurs is taken as σ_i here referred to as the instantaneous or static strength of the material. For stress levels lower than σ_i, controlled crack growth is taken to occur. The crack growth will continue up to the time at which the crack becomes sufficiently large such that at the then existing stress level, fracture will occur, this being by the same basic mechanism that causes the fracture instability at σ_i. The central problem is to determine this time to failure under a given stress history.

Once again it is necessary to appeal to and rely upon the power law representation in order to proceed further. The crack growth rate is taken to be controlled by the kinetic power law form

$$\dot{a} = \frac{da}{dt} = \lambda \left(\sigma \sqrt{a} \right)^r \tag{20}$$

expressed in terms of the stress intensity factor of the central crack problem from classical fracture mechanics. The crack size and far field stress in (20) have general time dependence and where in (20) r is the power law exponent

and λ is a material parameter. Equation (20) is written in the form shown because the central crack problem is being considered here. The power law form (20) for the time dependent crack growth problem is sometimes called the Paris law in the context of cyclic fatigue. The advantages and significance of power law forms have already been discussed.

Separate the variables in (20) and formally integrate to get

$$\int_{a_0}^{a(t)} \frac{da}{a^{\frac{r}{2}}} = \lambda \int_0^t \sigma^r(\tau) d\tau \qquad (21)$$

where a_0 is the initial size of the crack. Performing the explicit integration gives

$$\left(\frac{a}{a_0}\right)^{-\frac{r}{2}+1} - 1 = \lambda\left(1 - \frac{r}{2}\right) a_0^{\frac{r}{2}-1} \int_0^t \sigma^r(\tau) d\tau \qquad (22)$$

Relation (22) gives the crack size a(t) as a function of time, for a given stress history. In effect, the history of the stress intensity factor is now considered as known. The time flow in (22) will be allowed to continue up to the point at which the crack becomes unstable, which will be considered next.

Now at the end of the lifetime, instantaneous, unstable failure occurs. At this time of failure, $t=t_f$, the fracture condition is specified by the critical value of the stress intensity factor, thus

$$\left[\sqrt{a(t)}\sigma(t)\right]_{t=t_f} = \sqrt{a_0}\,\sigma_i \qquad (23)$$

Relation (23) provides the key to the present approach. The right hand side of (23) is the critical stress intensity factor for the instantaneous static strength while the left hand side is necessarily the same critical stress intensity factor but at the end of the lifetime under a load less than σ_i but a crack size greater than a_0.   At $t = t_f$ rewrite (23) as

$$\frac{a}{a_0} = \left(\frac{\sigma_i}{\sigma}\right)^2 \qquad (24)$$

Substitute (24) into (22) to obtain

$$\left(\frac{\sigma_i}{\sigma}\right)^{2-r} - 1 = \lambda\left(1 - \frac{r}{2}\right)a_0^{\frac{r}{2}-1}\int_0^{t_f}\sigma^r(\tau)d\tau \qquad (25)$$

Relation (25) can be put into nondimensional form. Let stress be nondimensionalized by the instantaneous static strength as

$$\tilde{\sigma} = \frac{\sigma}{\sigma_i} \qquad (26)$$

and let time be nondimensionalized as

$$\tilde{t} = \frac{t}{t_1} \qquad (27)$$

where

$$t_1 = \frac{a_0^{1-\frac{r}{2}}\sigma_i^{-r}}{\lambda\left(\frac{r}{2}-1\right)} \qquad (28)$$

With (26)-(28) the lifetime relation (25) becomes

$$1 - \tilde{\sigma}^{r-2}\left(\tilde{t}_f\right) = \int_0^{\tilde{t}_f}\tilde{\sigma}^r(\tau)d\tau \qquad (29)$$

Finally, drop the $\tilde{t}_f$ notation in (29) where $\tilde{t}$ will be understood to be the lifetime, and then (29) is

$$\frac{1}{1-\tilde{\sigma}^{r-2}(\tilde{t})}\int_0^{\tilde{t}}\tilde{\sigma}^r(\tau)d\tau=1 \tag{30}$$

This is the final form of the general lifetime criterion. Specific cases follow from (30) when the stress history is specified.

For a given stress history, $\tilde{\sigma}(\tau)$, the form (30) determines the lifetime $\tilde{t}$, to failure. The parameter t_1 in the nondimensionaized time (27) will be treated as a single free parameter, rather than using (28). The time parameter t_1 is effectively substituted for the parameter λ in (28). Parameter t_1 is accommodated by shifts along the log time axis. Relation (30) only contains two parameters, the power law exponent, r, and the time shift parameter, t_1, considering the instantaneous static strength, σ_i, to be known.

Consider the creep rupture condition which is specified by constant stress. From (30) the lifetime is found to be

$$\tilde{t}_c=\frac{1}{\tilde{\sigma}^r}-\frac{1}{\tilde{\sigma}^2} \tag{31}$$

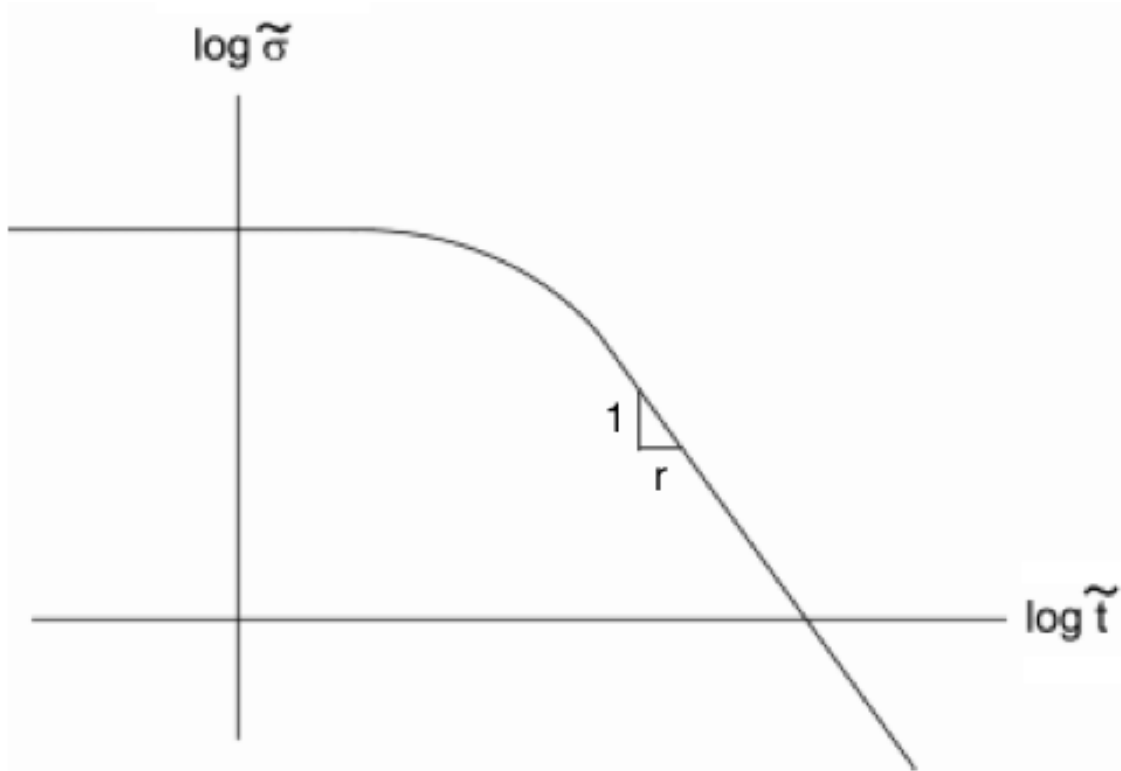The creep rupture lifetime result is shown schematically in Fig. 3.

Fig. 3   Creep rupture

The short time range is that of the instantaneous static strength asymptote and the long time range is the power law asymptote. The comparison of the creep rupture theoretical prediction with some experimental data is shown in Fig. 4.
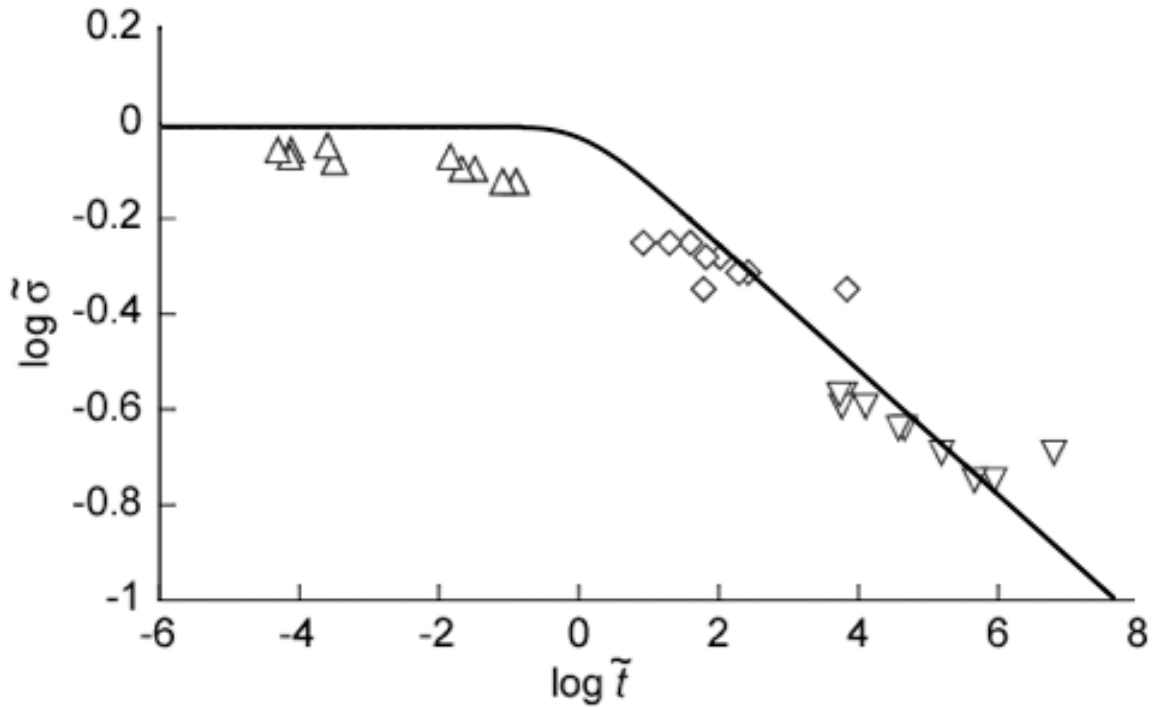
Fig. 4 Creep rupture (31) and data

The data is for a carbon fiber - vinyl ester resin laminate. The data is from Christensen and Miyano [6] which gives references to the earlier experimental work. It is seen that the data rounds the fairly sharp corner of the theoretical prediction. Probably this would occur in all applications to complex systems. An empirical form is given in Ref. [6] that fits the data quite closely

Constant Strain Rate

As an example of deterministic life prediction for a non-constant loading history, consider the case of constant strain rate. Since the bulk material is perfectly elastic, the case of constant strain rate is the same as constant stress rate then specified by

$$\sigma(\tau) = \beta\tau \tag{32}$$

Write the lifetime criterion (30) in dimensional form using (26) and (27) as

$$\frac{1}{\sigma_i^2 t_1 \left[\sigma_i^{r-2} - \sigma^{r-2}(t) \right]} \int_0^t \sigma^r(\tau) d\tau = 1 \qquad (33)$$

Substituting (32) into (33) and carrying out the integration gives

$$\beta^r t^{r+1} + (r+1)\beta^{r-2}\sigma_i^2 t_1 t^{r-2} - (r+1)t_1\sigma_i^r = 0 \qquad (34)$$

In general this is a high order polynomial to be solved for the time to failure, t. Rather than doing that directly, it is advantageous to eliminate the stress rate value β in favor of the stress at failure. Using (32) the write β as

$$\beta = \frac{\sigma_f}{t_f} \qquad (35)$$

where σ_f is the stress at the failure time $t = t_f$. With (35) then (34) becomes

$$\sigma_f^r t_f + (r+1)\sigma_i^2 t_1 \sigma_f^{r-2} - (r+1)t_1\sigma_i^r = 0 \qquad (36)$$

This relation directly gives the time to failure as

$$\frac{t_f}{t_1} = (r+1)\left[\left(\frac{\sigma_i}{\sigma_f} \right)^r - \left(\frac{\sigma_i}{\sigma_f} \right)^2 \right] \qquad (37)$$

Take the logarithm of (37) to find

$$\log\frac{t_f}{t_1} = \log\left[\left(\frac{\sigma_i}{\sigma_f}\right)^r - \left(\frac{\sigma_i}{\sigma_f}\right)^2\right] + \log(r+1) \qquad (38)$$

From (38) it is seen that the constant strain rate lifetime is the same as the creep rupture result (31) but shifted by the amount

$$\log(r+1)$$

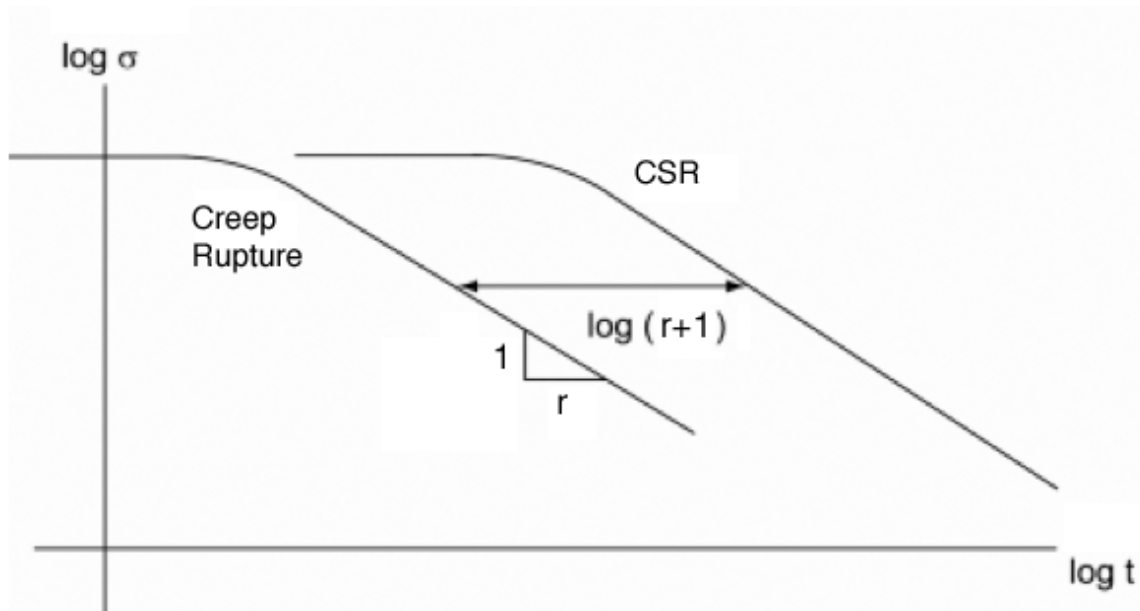Along the log time scale this shifting property is shown in Fig. 5.

Fig. 5   Constant strain rate shifting property

The shifting property has been experimentally verified using data from a polymeric fiber composite laminate, Fig. 6, from Ref. [5].
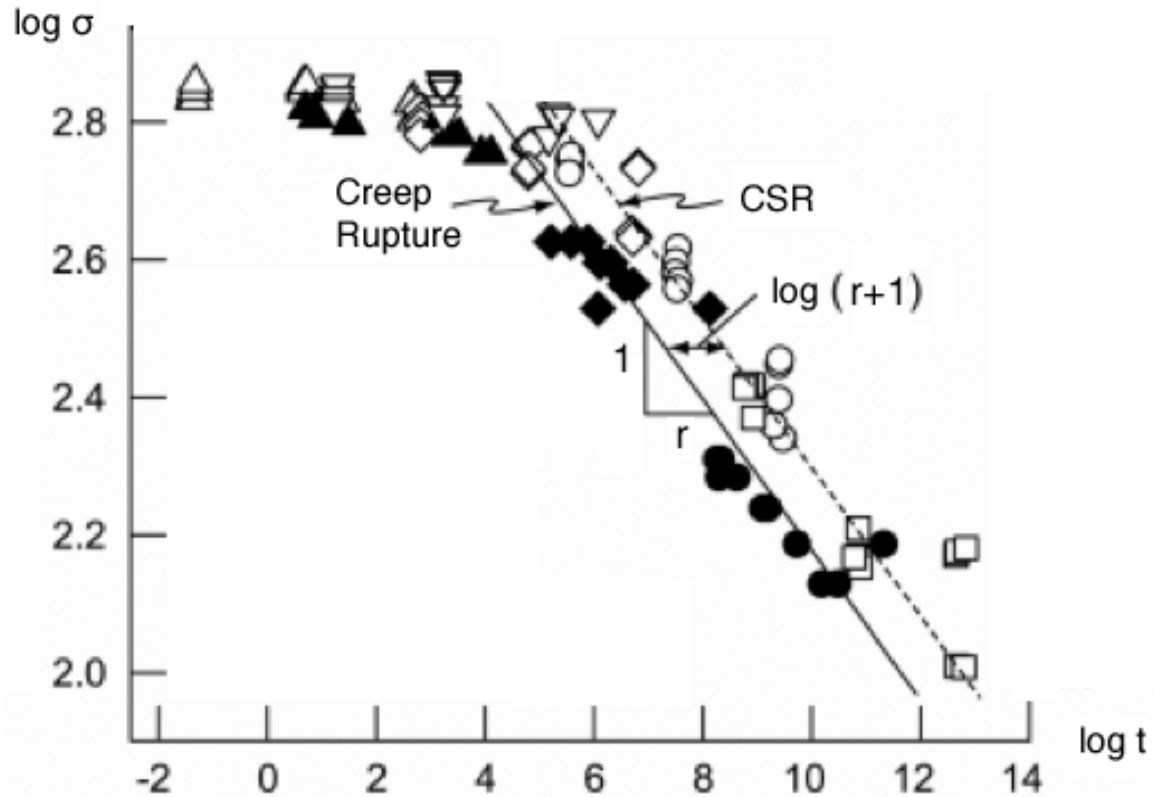
Fig. 6   Constant strain rate prediction and data

Probabilistic Generalization of Lifetime Theory and Its Evaluation

   When one first sees the raw data from creep rupture testing, the problem looks hopeless. There is no apparent or discernable order to the extremely scattered data. Yet when the proper probabilistic treatment is applied, order does indeed begin to emerge.

   From the previous deterministic formulation now let the instantaneous static strength, σ_i, be characterized as a random variable with a specific distribution function. This physical effect results from the inherent scatter in the initial flaw sizes or weaknesses in the virgin material.

   Let deterministic σ_i be specified through a Weibull distribution where the complement of the CDF, $F(\sigma)$, is given by

$$F(\sigma) = e^{-\left(\frac{\sigma}{\sigma_s}\right)^m} \tag{39}$$

where m is the shape parameter and σ_s is the scale parameter. Now let

$$F(\sigma) = 1 - k \tag{40}$$

where k is the quantile of failure for the instantaneous static strength with k=0 meaning no failures. It follows that

$$\sigma_k = \sigma_s \left[-Ln(1-k) \right]^{\frac{1}{m}} \tag{41}$$

where now σ_k is the probabilistic form for σ_i.

The deterministic creep rupture result (31) is rewritten here as

$$\tilde{t} = \left(\frac{\sigma_i}{\sigma} \right)^r - \left(\frac{\sigma_i}{\sigma} \right)^2 \tag{42}$$

Using (41) the probabilistic generalization of (42) is given by

$$\tilde{t} = \frac{\left[-Ln(1-k) \right]^{\frac{r}{m}}}{\left(\frac{\sigma}{\sigma_s} \right)^r} - \frac{\left[-Ln(1-k) \right]^{\frac{2}{m}}}{\left(\frac{\sigma}{\sigma_s} \right)^2} \tag{43}$$

Write (43) in symbolic form as

$$\tilde{t} = \hat{f} \left(\frac{\sigma}{\hat{\phi}(k)} \right) \tag{44}$$

where

$$\hat{\phi}(k) = \sigma_s \left[-Ln(1-k) \right]^{\frac{1}{m}}$$ (45)

and

$$\hat{f}(x) = \frac{1}{x^r} - \frac{1}{x^2}$$ (46)

The results (44)-(46) give the probabilistic time to failure for the creep rupture condition as a function of the Weibull shape and scale parameters of the instantaneous static strength, the power law exponent r, and the specified quantile of failure k. The same probabilistic generalization can be applied to the lifetime form (30) for any prescribed stress history.

Now examine the power law range of behavior for creep rupture. From (43) this range is specified by

$$\tilde{t} = \left(\frac{\sigma_s}{\sigma} \right)^r \left[-Ln(1-k) \right]^{\frac{r}{m}}$$ (47)

Solve (47) for (1-k) to get the complement to the lifetime CDF as

$$1 - k = e^{\left[-\left[\frac{t}{t_1 \left(\frac{\sigma_s}{\sigma} \right)^r} \right]^{\frac{m}{r}} \right]}$$ (48)

From (48) it is seen that the probabilistic lifetime in the power law range is Weibully distributed with

$$\frac{m}{r} - lifetime\ shape\ parameter \qquad (49)$$

$$t_1\left(\frac{\sigma_s}{\sigma}\right)^r - lifetime\ scale\ parameter \qquad (50)$$

The results (49) and (50) not only are unusually compact but extraordinarily comprehensive. The lifetime probabilistic behavior is completely specified by the instantaneous static strength Weibull parameters m and σ_s and the slope of the lifetime envelopes in the power law range, 1/r. Parameter t_1 is most easily found directly from lifetime data. The significance of these results will be examined next.

In the 1970's, 80's and 90's a nearly unique program of creep rupture testing was conducted at Lawrence Livermore National Laboratory. T. T. Chiao formulated and managed a very large program of testing involving many hundreds of specimens. Some of them were maintained for as long as many years under load before failure. The testing specimens were thin strands of unidirectional fibers impregnated with epoxy resins. The three fiber types were carbon, aramid, and glass. Dead loads were applied at different loading levels and timing devices recorded failure. It was a meticulously planned program with dedicated long term follow through. The testing results are known as the LLNL data bases for creep rupture.

These creep rupture data bases were used to evaluate this highly idealized probabilistic lifetime theory. In particular, using the result (49) the predicted lifetime shape parameters were compared with the measured lifetime shape parameters. There could hardly be a more critical evaluation than this. The lifetime failure data are spread over many decades of time while the instantaneous static strength data are tightly grouped around a mean stress level. The evaluation thus tests both the Weibull distribution hypothesis for static strength and lifetime as well as the kinetic crack life prediction theory. All of the data sets were found to be satisfactorily modeled by Weibull distributions and the shape and scale parameters were determined by the method of maximum likelihood. All exhibited the expected power law ranges of behavior.

The especially important case of AS-4 carbon-epoxy was analyzed by Christensen and Glaser [7].   The Kevlar aramid-epoxy system was also analyzed by Christensen and Glaser [8]. Finally, the S glass-epoxy system was analyzed by Glaser, Christensen and Chiao [9]. The basic theory was developed in the 1970's and 80's by Christensen, in other publications that were the forerunners of much of what is presented in this sub-section and the previous one. The notation used in all of these references was a little different from that used here, but they all are compatible. The notation here is simpler.

The probabilistic data reductions and evaluations directly from the above three references are summarized as follows:

Carbon – Epoxy

Power law exponent r	72.2	Measured
Static strength shape parameter	16.0	Measured
Lifetime shape parameter	0.191	**Measured**
Lifetime shape parameter	0.222	**Predicted**

Aramid – Epoxy

Power law exponent r	45.6	Measured
Static strength shape parameter	46.3	Measured
Lifetime shape parameter	0.960	**Measured**
Lifetime shape parameter	1.02	**Predicted**

Glass – Epoxy

Power law exponent r	32.8	Measured
Static strength shape parameter	30.6	Measured
Lifetime shape parameter	0.852	**Measured**
Lifetime shape parameter	0.933	**Predicted**

Table 1   Creep rupture data and evaluation of theory

The term "Measured" means determined from the data bases, while "Predicted" refers to the theoretical result (49) coming from (48) for the shape parameter of lifetime.

Thus the enormously broad distributions of the creep rupture lifetimes are successfully predicted by the extremely narrow distributions of the static strength and the slopes of the probability quantiles in the lifetime power law ranges. It is now easy to see how designing for lifetime probabilities becomes a practical proposition using only a relatively few specimens for testing. Considering the importance and complexity of the problem, the clarity of this solution is something quite special. Unifying results such as these do not come along very often in such difficult fields.

Now the Weibull specific and power law specific results (44)-(46) will be generalized to any corresponding form by taking the non-dimensional creep rupture time to failure $\tilde{t}$ as

$$\tilde{t} = f\left(\frac{\sigma}{\phi(k)}\right) \tag{51}$$

where

$$\phi(k) - not\ necessarily\ of\ Weibull\ form$$

and

$$f\left(\frac{\sigma}{\phi(k)}\right) - not\ necessarily\ of\ power\ law\ form$$

The Weibull and power law forms of $\varphi(\ )$ and $f(\ )$ are given by (45) and (46), but now for the more general forms all that is required is that $\varphi(k)$ be any form controlling the probabilistic instantaneous static strength and σ and $\varphi(k)$ enter $f(\ )$ only in the combination $f(\frac{\sigma}{\phi(k)})$ and that $f(\frac{\sigma}{\phi(k)}) = 0$ when

$\frac{\sigma}{\phi(k)}$=1, compare with (46). Thus the forms of φ(k) and f( ) remain quite general.

Express the form (51) in terms of log variables. This gives (51) as

$$\log \tilde{t} = g\left(\log\left(\frac{\sigma}{\varphi(k)} \right) \right) \qquad (52)$$

where the function g(log x) is found from the form of f(x) in (51). Rewrite (52) as

$$\log \tilde{t} = g\big(\log(\sigma) - \log\phi(k)\big) \qquad (53)$$

The result (53) on log σ versus log t scales has the form shown in Fig. 7 for different values of the quantile of failure, k.
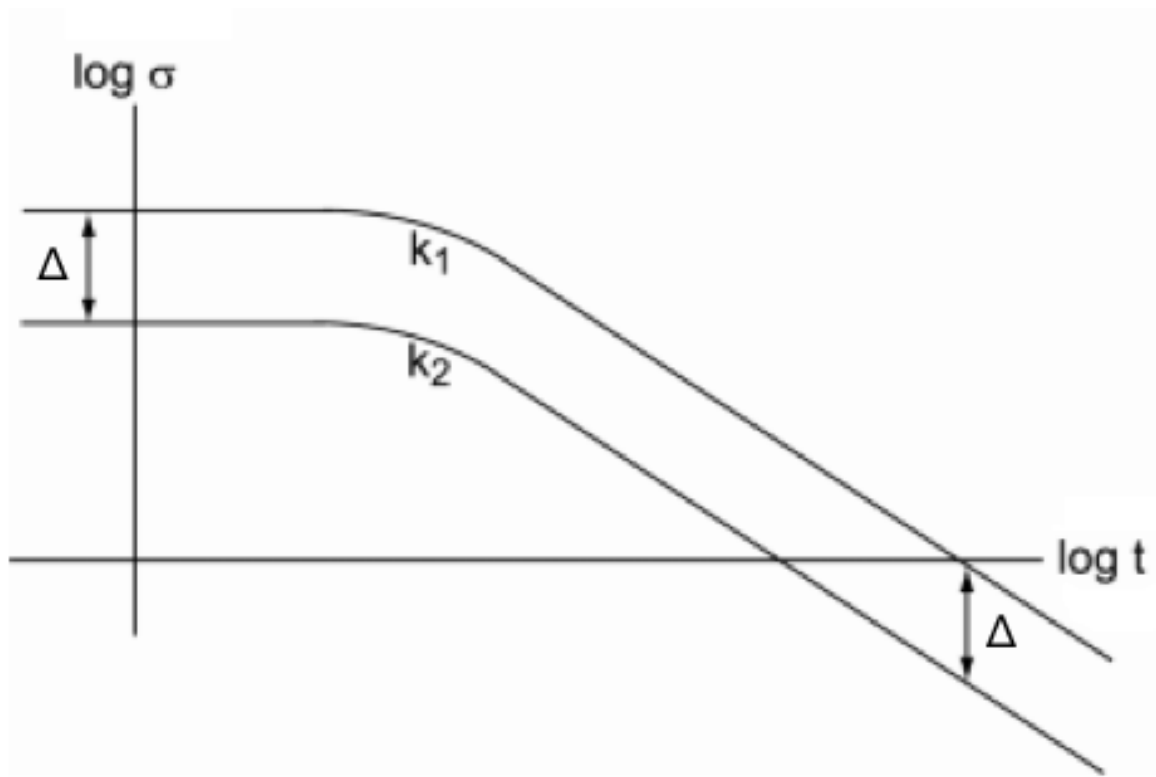

Fig. 7   Vertical shifting property

The result (53), as shown in Fig. 7, reveals that the different probability levels of failure have the form of a single, master curve that is shifted vertically along the log σ axis. This unexpected and extremely simplifying type of behavior was first deduced by Christensen [10].

Relative to the vertical shifting behavior shown in Fig. 7, twenty specimens of unidirectional carbon fiber – vinyl ester composites were tested in the constant strain rate condition at elevated temperature to simulate the time dependent case and also tested for instantaneous static strength. The results are shown in Fig. 8, Christensen and Miyano [5].
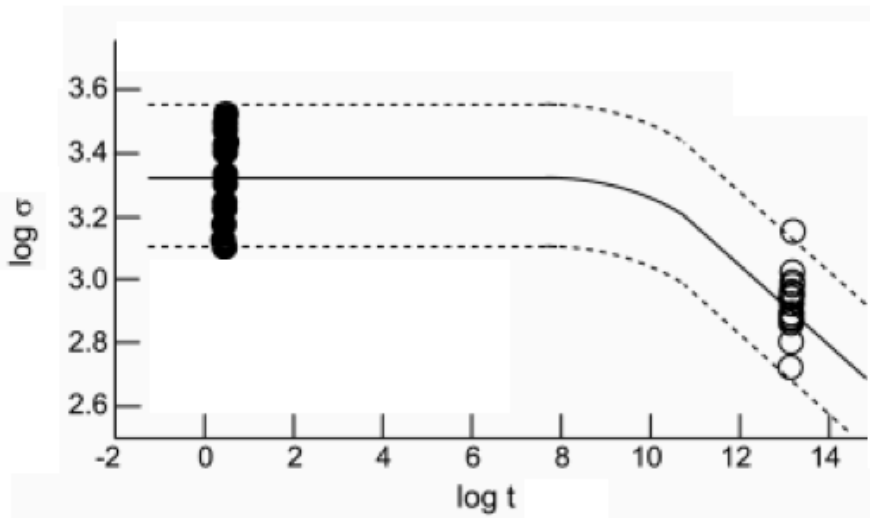
(a) Data of tensile CSR
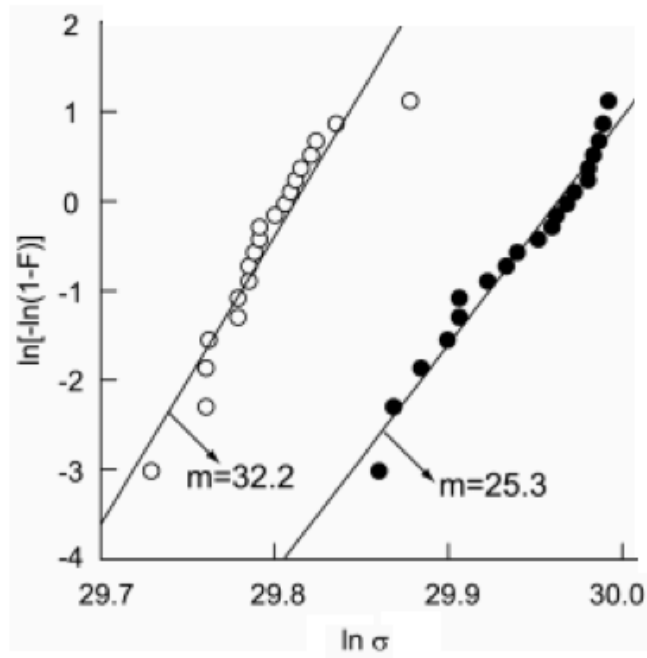
(b) Weibull probability

Fig. 8   Weibull probability for vertical shifting

The vertical shift result was evaluated as shown. The results were found to be of Weibull distribution, Fig. 8, with the two shape parameter of m=32.2 and m=25.3. While these are not quite as close as would be ideal, they do show consistency with the vertical shift behavior. That they are both of the Weibull type is also consistent with the vertical shift.

This completes the coverage here of primarily using the Weibull distribution with the cases of probabilistic failure. Successful though it is, the Weibull distribution is not the answer to all problems in the macroscopic domain. Sometimes a particular Weibull distribution satisfactorily covers a certain range, but then another distribution seems to explain the more extreme ranges and cases. This secondary distribution could be a different Weibull distribution such as the exponential distribution, or it could be an entirely different type of distribution. Even in cases such as the later case, the Weibull distribution still seems to cover the main aspects of the probabilistic failure behavior.

Needless to say this has been and continues to be an active field. There is a broad and important literature in the field in addition to the book by Rinne [4] and many other books, as well as the references already given. Bazant and colleagues [11-13] have provided a comprehensive treatment of probabilistic failure based upon theories of behavior at the nanoscale. Phoenix and colleagues [14-16] have provided longstanding treatments of probabilistic failure, primarily for composites. Miyano, Nakada and colleagues [17-19] have provided a very extensive literature of testing results including probabilistic behavior obtained by the method of accelerated testing using elevated temperatures for polymeric based composite fiber systems.

The problem of cyclic fatigue in metals admits an analogy with the present problem of creep rupture in polymers. Taking (20) as da/dn rather than da/dt, with n, the number of cycles, as the measure of duration, then all of the statistical forms and conclusions found here for creep rupture have a one to one counterpart in the cyclic fatigue area. Even though the controlling properties (power law exponent, shape and scale parameters) could and would be very different for fatigue than for creep rupture and very different for different materials, nevertheless the same mathematical formalism applies in both cases. Necessarily this would be subject to

independent experimental verification in any particular case, such as metal fatigue.

A technical area closely related to that of creep rupture and cyclic fatigue is that of size dependence. The disparity between the size of test specimens and the size of full scale applications could cause divergences, perhaps large divergences. This is a well known effect, presumably about as difficult to treat as that of life prediction with no size dependence. Some research articles on size/scale dependence are those of Barenblatt [20], Barenblatt and Botvina [21] and Ritchie [22]. The articles by Bazant et al mentioned above give full account of scale dependence, as integrated with life prediction. In the present probabilistic context, size dependence could possibly enter the theory through several of the physical variables, including the shape and scale parameters of the instantaneous static strength as well as the power law form for the kinetic crack growth, especially the exponent in the power law. Many other size dependent effects could also be involved.

Future work here will take up the problem of size scaling for the failure and life prediction of materials.

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May 19th, 2012