XV. FAILURE MECHANICS: THE COORDINATION BETWEEN ELASTICITY THEORY AND FAILURE THEORY

Status of Failure

This work establishes an important relationship between elasticity theory and failure theory for three dimensional homogeneous and isotropic materials. At first exposure, one could reasonably question how there could possibly be any such connection between the elasticity formulation and that of failure. For well over a hundred years mainstream thinking has held that elastic behavior at increasing levels of stress yields into plastic behavior and that then progresses until some critical plasticity limit condition inevitably requires final and complete rupture. Unless the failure was totally brittle, failure has nearly universally been perceived as the ultimate extreme and conclusion of plastic deformation. It is so reported in virtually all tables of properties for specific materials.

A newly developed failure theory has challenged that longstanding, conventional view. This new theory, Christensen [1], is based upon two physical postulates. The first is that failure represents the termination of elastic behavior, not plastic behavior. This is obviously true for brittle materials, but for elastic-plastic behavior, failure is still taken to most fundamentally signify the cessation of the previous elastic behavior. The possible occurrence of plasticity simply represents a more complex transition from the elastic state to the failure state than occurs with brittle materials.

The common elastic-perfectly plastic idealization convincingly illustrates the futility of trying to use plasticity to directly and solely define failure in terms of an actual rupture. Other contrary

examples also are apparent, but have not been sufficiently well appreciated.

This new approach for failure theory in no way detracts from the meaning, the value, and the usefulness of plasticity theory. There are large classes of problems involving highly nonlinear and permanent deformation requiring plasticity treatments. It simply asserts that the general definition of three dimensional failure criteria must be viewed by means other than through the lens of ultimate plastic deformation.

The second and equally important key to the new theory of failure is the hypothesis that the strengths ratio T/C covers the full range of all possible behaviors for homogeneous and isotropic materials. Symbols T and C are the failure stresses in uniaxial tension and compression. Thus

$$0 \leq \frac{T}{C} \leq 1$$

provides the failure descriptor for all failure types and classes.

There is an important corollary and consequence to this ratio of the uniaxial strengths as representing the full and complete spectrum of possible failure behaviors. It follows from this hypothesis that the general theory must be of a two property form, not the three or four or more parameters empirical forms that are often thought and argued to be necessary.

These two physical hypotheses are sufficient to guide the way to the mathematical development of the general, complete, and self contained theory of materials failure, [1], for full density, homogeneous and isotropic materials.

After stating the controlling forms for the new failure formalism and for linear elasticity, their explicit relationship will be closely examined in the next two sections. It will be shown that there is a type of coordination or coupling between the two formalisms. This coordination concerns the differentiation between ductile and brittle types of failures. The elasticity part of the combination reveals new and independent information on this important subject. The enabling development that makes this possible is the renormalization of the classical theory of elasticity, presented here for the first time.

Finally, the last section considers the unification of elasticity theory and failure theory and treats the overall discipline as that of failure mechanics. This failure mechanics is for homogeneous and isotropic materials and it is completely separate from and complementary to the field of fracture mechanics. The overall coverage and significance of all of the historical mechanics related disciplines, including these two, are given a final survey and assessment.

Note on References and Content: All references to previous supporting work will be through the new book "The Theory of Materials Failure", Ref. [1]. However, the results in this website are completely compatible with the corresponding results in the book. The advantage of the book lies in its more extensive treatments of background and history, for further details into the difficult derivations, and for interpretations. Although the following developments will likely seem somewhat complex and far from routine, they will be rewarding for the new information and insights that they offer into materials failure.

Failure Theory and Renormalized Elasticity Theory

The constitutive relations for linear, isotropic elasticity theory will be recalled for use here in one of its many but equivalent

forms. The most obvious and natural form is that of the expression of the elastic energy partitioned into dilatational and distortional terms and calibrated by the corresponding properties. This can be done with either stresses or strains. The account here will begin with stresses and then later return to strains. This form of elastic energy is given by

$$U = \frac{1}{2k}\left(\frac{\sigma_{ii}}{3}\right)^2 + \frac{1}{4\mu} s_{ij} s_{ij} \tag{1}$$

where k is the bulk modulus and μ is the shear modulus. The deviatoric stresses and strains are defined by

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk}$$

$$e_{ij} = \varepsilon_{ij} - \frac{\delta_{ij}}{3}\varepsilon_{kk} \tag{2}$$

The failure theory given by Christensen [1] has the basic criterion developed by the method of polynomial invariants as

$$\left(1 - \frac{T}{C}\right)\hat{\sigma}_{ii} + \frac{3}{2}\hat{s}_{ij}\hat{s}_{ij} \le \frac{T}{C} \tag{3}$$

where the stress is nondimensionalized by the compressive strength

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{C} \tag{4}$$

and where T and C are the failure stresses with the brittle to ductile limits and range specified by

$$0 \le \frac{T}{C} \le 1 \tag{5}$$

The failure criterion (3) is and must be supplemented by a competitive fracture criterion given by

$$\hat{\sigma}_1 \le \frac{T}{C} \qquad if \qquad \frac{T}{C} \le \frac{1}{2} \tag{6}$$

where σ_1 is the maximum principal stress. See Ref. [1] for the full background, motivation, the derivation, and for the interpretation of this new theory of failure.

All the attention here will be upon the polynomial invariants failure criterion (3). Nothing will be done here that involves or would be superseded by the fracture criterion (6).

The elastic energy (1) and the failure criterion (3) are from completely independent derivations with each receiving an independent polynomial invariants type of expansion in the invariants of the stress tensor.

It will be helpful to express the elastic energy (1) in the alternative and less obvious, hybrid form using Young's modulus, E, and Poisson's ratio, ν, but still retaining the dilatational and distortional source terms. This converts (1) to

$$U = \frac{1}{2E}\left[\frac{(1-2\nu)}{3}\sigma^2{}_{ii} + (1+\nu)s_{ij}s_{ij}\right] \tag{7}$$

where it is clear that ν must have the limits

$$-1 \le \nu \le \frac{1}{2} \tag{8}$$

Hencforth E will be referred to as the elastic modulus.

Compare the elastic energy (7) and the failure form (3). They are of somewhat similar forms but with some distinct differences, especially in the quadratic dependence on σ_{ii} in (7) but the linear dependence on it in (3). These two forms (3) and (7) appear slightly more similar when (7) is rewritten as

$$U = \frac{(1+\nu)}{2E}\left[\frac{1}{3}\left(\frac{1-2\nu}{1+\nu}\right)\sigma_{ii}^2 + s_{ij}s_{ij}\right] \tag{9}$$

Using the limits for ν in (8) and the limits for T/C in (5) then the range of the coefficient of $\hat{\sigma}_{ii}$ in (3) is from 0 to 1 while the range of the coefficient of σ_{ii}^2 in (9) is from 0 to ∞. In a sense the elasticity theory based on (7) is formalized such that the governing property, ν, is normalized at the value of ν = -1. It could be helpful if the elasticity theory were renormalized with a different nondimensional variable. Renormalization is often used in physics for special utility or for new insights. The renormalization will be centered on ν = 0. All properties will be taken to be unchanged at and only at ν = 0.

As the first step in this direction, recognize that it is only the positive values of ν that are physically realistic, $0 \le \nu \le ½$. Negative values for ν are never found for homogeneous and isotropic materials. There is a very good reason for this behavior and it exists at the nano-scale. This restriction will be detailed

later. But for now, it is henceforth taken that only positive values of ν carry and convey physical reality.

It would furthermore be advantageous if the elasticity theory could be renormalized such that ν with $0 \le \nu \le ½$ renormalizes to a variable that ranges from 0 to 1 the same as with T/C in (5).

So the renormalization of elasticity theory is the main objective here. To this end, compare the coefficients of σ_{ii}^2 and $\hat{\sigma}_{ii}$ in (9) and (3). To bring them into closer alignment take

$$\left(\frac{1-2\nu}{1+\nu}\right) = 1-\nu^* \tag{10}$$

where ν^* is a tentative new nondimensional property. Solving (10) for ν^* gives

$$\nu^* = \frac{3\nu}{1+\nu} \tag{11}$$

Thus ν^* is found directly and solely from ν.

Over the range of ν in (8) it follows from (11) that ν^* has the range

$$-\infty \le \nu^* \le 1 \tag{12}$$

But over the physically relevant and physically acceptable range of ν as

$$0 \le \nu \le \frac{1}{2} \tag{13}$$

then from (11) the corresponding range on ν^* is

$$0 \le \nu^* \le 1 \tag{14}$$

The range of ν^* in (14) is of the required form needed to move forward. The new property ν^* in place of ν is the renormalized Poisson's ratio.

Correspondingly take the renormalized elastic modulus as

$$E^* = \frac{E}{1+\nu} \tag{15}$$

Then the renormalized constitutive relation for linear elasticity from (9), (11), and (15) is given by

$$U = \frac{1}{2E^*}\left[\frac{\left(1-\nu^*\right)}{3}\sigma_{ii}^2 + s_{ij}s_{ij}\right] \tag{16}$$

where ν^* has the general range of (12) but the physically restricted and realistic range of (14). Renormalized modulus E^* is always positive.

From this point onward the range of ν^* will always be taken as (14) even though it applies over the full range of (12). Next the consequent elastic forms that follow from (16) will be derived.

The classical two Lame′ constants and the bulk modulus are given by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)} \tag{17}$$

$$k = \frac{E}{3(1-2\nu)}$$

These properties in terms of E^* and ν^* are then given by

$$\lambda = \frac{\nu^* E^*}{3(1-\nu^*)}$$

$$\mu = \frac{E^*}{2} \tag{18}$$

$$k = \frac{E^*}{3(1-\nu^*)}$$

The variational result

$$\varepsilon_{ij} = \frac{\partial U}{\partial \sigma_{ij}} \tag{19}$$

gives the consequent strain-stress relations from (16) as

$$\varepsilon_{ij} = \frac{1}{E^*}\left[\sigma_{ij} - \frac{\nu^*}{3}\delta_{ij}\sigma_{kk}\right] \tag{20}$$

The other two complementary forms are found to be the energy form in terms of strains as

$$U = \frac{E^*}{2}\left[\frac{1}{3\left(1-\nu^*\right)}\varepsilon_{kk}^2 + e_{ij}e_{ij}\right] \tag{21}$$

and the stress-strain relations as

$$\sigma_{ij} = E^*\left[\varepsilon_{ij} + \frac{\nu^*}{3\left(1-\nu^*\right)}\delta_{ij}\varepsilon_{kk}\right] \tag{22}$$

Taking mean normal stresses and strains as

$$\sigma_M = \frac{\sigma_{ii}}{3}$$

$$\varepsilon_M = \frac{\varepsilon_{ii}}{3} \tag{23}$$

then gives the slightly different forms of the four elastic constitutive relations as being

$$U = \frac{1}{2E^*}\left[3\left(1-\nu^*\right)\sigma_M^2 + s_{ij}s_{ij} \right] \tag{24}$$

$$U = \frac{E^*}{2}\left[\frac{3}{\left(1-\nu^*\right)}\varepsilon_M^2 + e_{ij}e_{ij} \right] \tag{25}$$

$$\varepsilon_{ij} = \frac{1}{E^*}\left[\sigma_{ij} - \nu^*\delta_{ij}\sigma_M \right] \tag{26}$$

$$\sigma_{ij} = E^*\left[\varepsilon_{ij} + \frac{\nu^*}{\left(1-\nu^*\right)}\delta_{ij}\varepsilon_M \right] \tag{27}$$

where E^* and ν^* are defined by (15) and (11), and subscript M is not an index. E^* and ν^* could be determined other than from the experiments implied in (11) and (15). This will be outlined in the next section.

These two groups of four relations (16), (20), (21), and (22) or (24)-(27) specify the renormalized forms of linear elasticity. Any one of the four relations serves to define the other three. These four simple forms remain as the same classical theory of elasticity, there are no approximations. Furthermore, these tensor results are probably the most compact, and most concise statements of the linear elastic constitutive relations. The advantage of expressing energy and strains in terms of stresses, (24) and (26), is apparent.

The four constitutive forms also reveal that the two properties E and ν, with subsequent E^* and ν^*, are highly preferable and more basic than the pairing of μ and k. The properties μ and k in these four forms lack the same symmetry and simplicity and in some

cases are ambiguous because of unlimited behaviors in limiting cases.

There likely is no advantage in using this new formalism for two-dimensional or one-dimensional conditions, nor for numerical computations. It is purely for the three dimensional formulation that is needed for basic theory purposes, especially in relation to failure conditions. It is somewhat surprising that this renormalized formulation of elasticity theory was not recognized much earlier in the past 100-150 years of intensive study.

From relation (11) some corresponding values of ν and ν^* are given by

ν	ν^*
0	0
1/8	1/3
1/5	1/2
1/3	3/4
1/2	1

Inverting (11) and (15) gives ν and E expressed in terms of ν^* and E^* as

$$\nu = \frac{\nu^*}{3-\nu^*} \tag{28}$$

$$E = \frac{3E^*}{3-\nu^*} \tag{29}$$

Some further relationships are as follows

$$E^* = E \qquad at \qquad \nu = \nu^* = 0$$

$$E^* = \frac{2}{3}E \qquad at \qquad \nu = \frac{1}{2}, \quad \nu^* = 1 \tag{30}$$

Although E^* and E are about the same size there is a significant difference and the proper property is E^* not E for the three dimensional calibration of elastic constitutive equations.

E^* is the universal and dimensional property expressing a materials general capability to resist deformation while bearing load. Property ν^* is the universal and nondimensional new property with what will be found to be a deep and insightful interpretation of what must be happening at the atomic scale that controls the balance of the energy sources involving the dilatational versus the distortional resistance.

Both the elastic energy form (16) and the failure form (3) involve a single dimensional property and a single nondimensional property. It follows that the limits of the nondimensional property in (16) associates with either a ductile or a brittle failure condition in uniaxial tension as given by

$$\nu^* = 0 \qquad \textit{Totally Brittle}$$

$$\nu^* = 1 \qquad \textit{Perfectly Ductile} \tag{31}$$

The basis for relations (31) is as follows. At $\nu^* = 0$ the constitutive relation (26) shows that the stress and strain tensors are co-axial, there is no Poisson's ratio effect. This means that there is no coupling between the stress components. Then the related failure behavior corresponds to the maximum normal stress

criterion of Lame′ and Rankine, which has always been interpreted as that of brittle fracture. Therefore this limit of $\nu^* = 0$ is taken as that of the totally brittle state.

At the other limit of $\nu^* = 1$, relation (24) shows that there is no dilatational contribution to the energy and (25) shows that there is no dilatation, $\varepsilon_M = 0$. The deformation is purely distortional. This is the state of incompressibility which usually corresponds to the behavior of elastomers and approached by extremely ductile metals. This is the state of perfect ductility.

Relations (31) give the first indication of a possible coupling behavior between the two very different theories of mechanical behavior, elastic energy and failure. The ductile versus brittle behavior as a function of T/C is fully explored in Ref. [1]. That the Poisson's ratio through ν^* also associates with brittle versus ductile behavior comes as an unexpected result from this derivation. However, it is not completely unexpected from a practical, observational point of view. Beryllium has a Poisson's ratio of virtually 0 and it is very brittle. Gold has a very large Poisson's ratio and it is very ductile. It is this possible relationship between ν^* and ductility that will be pursued further.

As a prelude to the further developments, it can now be rationalized that there could conceivably be some relationship between elasticity theory properties and failure theory properties, especially since failure is perceived as the termination of elastic behavior.

The Coupling Between Elasticity and Failure

As derived in Ref. [1] the state of uniaxial tension has a specific measure of the degree of ductility D through

$$D = \frac{T}{C} \qquad \textit{Uniaxial Tension} \tag{32}$$

This extraordinarily simple ductility measure goes from 0 to 1 meaning from the state of total brittleness to the state of perfect ductility. Similar although more complicated forms for the ductility measures can be derived for any stress state and are done so in [1]. But the interests here are solely with uniaxial tension since it is by far the most important special case, and it is universally used to calibrate comparative ductility levels for different materials. The ductile brittle transition for uniaxial tension in (32) is at T/C = 1/2. This also is explained and explored at length in [1] and will later be derived here by a different method.

In view of the ductility limits shown in (31), it follows that the renormalized elastic property ν^* may have some use as a continuous measure of the degree of ductility in uniaxial tension. To this end, take D^* as the possible measure of ductility in terms of ν^*. At this point it is only known from (31) that D^* as a function of ν^* has the limiting behaviors

$$\begin{aligned} D^*\left(\nu^*\right) &= 1 \qquad at \qquad \nu^* = 1 \\ D^*\left(\nu^*\right) &= 0 \qquad at \qquad \nu^* = 0 \end{aligned} \tag{33}$$

Let the possible ductility measure take the general power law form

$$D^*\left(\nu^*\right) = \left(\nu^*\right)^n, \quad n \geq 1 \tag{34}$$

The form, (34) satisfies the required limits in (33). The exponent n in (34) must be taken to satisfy some as yet unspecified but logical condition. This condition must involve compatibility with the observed behaviors of the entire spectrum of isotropic materials types.

At this point the elastic energy and the polynomial invariants failure criterion from (16) and (3) will be rewritten here for direct comparison with each other. These are

$$\frac{\left(1-\nu^{*}\right)}{3}\sigma_{ii}^{2}+s_{ij}s_{ij}=2E^{*}U \tag{35}$$

and

$$\left(1-\frac{T}{C}\right)\hat{\sigma}_{ii}+\frac{3}{2}\hat{s}_{ij}\hat{s}_{ij}\leq\frac{T}{C} \tag{36}$$

where the limits on ν^{*} are given by (14) and those on T/C by (5).

Note that all the forms and properties in the two relations (35) and (36) are either quadratic or linear in form. In accordance with these characteristics, the most likely values for exponent n in (34) are either n = 1 or n = 2, linear or quadratic.

From (36) it is seen that T/C in the first term is linear and this may therefore favor taking n = 1 in the ductility measure (34). On the other hand, it is also seen that T/C is associated with the linear term $\hat{\sigma}_{ii}$ in (36) while ν^{*} is associated with the quadratic term σ_{ii}^{2} in (35). This may favor taking n as being quadratic, n = 2.

More specific reasoning to ascertain the proper value for n is as follows. The ductile brittle transition in terms of T/C is at

T/C =1/2. This was developed by a general and elaborate methodology in Ref. [1]. A much simpler approach will be sufficient for the restricted case of uniaxial tension and will now be given.

For uniaxial stress, σ, the failure criterion (36) reduces to

$$\underset{\uparrow}{\left(1-\frac{T}{C}\right)\frac{\sigma}{C}} \quad + \quad \underset{\uparrow}{\left(\frac{\sigma}{C}\right)^2} \le \frac{T}{C} \tag{37}$$

Dilatational *Distortional*

For uniaxial tensile failure at σ = T and for equality in (37) it becomes

$$\left(1-\frac{T}{C}\right)+\left(\frac{T}{C}\right)=1 \tag{38}$$

The separate sources from dilatation and distortion are so labeled in (37). At the two limits, (38) becomes

$$At \quad \frac{T}{C}=1, \qquad \underset{\uparrow}{0+1=1} \tag{39}$$

Distortional

and

$$At \quad \frac{T}{C} = 0, \qquad 1 + 0 = 1$$

$$\uparrow \qquad (40)$$

Dilatational

At the first limit, (39) the distortion source dominates and the behavior is ductile. Of course plasticity behavior will ensue near the T/C = 1 end of the scale since the distortional source dominates. At the second limit, (40) the dilatation source dominates and the failure behavior is brittle.

At the value of T/C = 1/2, (38) becomes

$$At \quad \frac{T}{C} = \frac{1}{2}, \qquad \frac{1}{2} + \frac{1}{2} = 1 \qquad (41)$$

and the dilatational source and the distortional source are in balance and this is taken as the ductile brittle transition. Thus the value T/C = 1/2 denotes the ductile brittle transition in uniaxial tension for this failure criterion. This is in accordance with the much more general method from [1].

Just as T/C = 1/2 gives the ductile brittle transition, the corresponding ductile brittle transition in terms of ν^* will be taken as mid-way between the limits for D^* of 0 and 1, namely at $D^* = 1/2$. The two candidate measures for D^* then give the respective ductile brittle transition as occurring at

$$n = 1, \qquad D^* = \nu^* = \frac{1}{2} \qquad (42)$$

or at

$$n = 2, \qquad D^* = \left(\nu^*\right)^2 = \frac{1}{2}, \quad \nu^* = \frac{1}{\sqrt{2}} \tag{43}$$

From relation (28) it further follows that the conceivable ductile brittle transition occurs at

$$n = 1, \qquad \nu = \frac{1}{5} \tag{44}$$

or at

$$n = 2, \qquad \nu = \frac{1}{3\sqrt{2} - 1} = 0.3084 \tag{45}$$

The n = 1 prediction of the ductile brittle transition as occurring at ν= 1/5 is completely unrealistic. Materials such as carbon (diamond) and silicon are near this value of Poisson's ratio and they are distinctly brittle in failure behavior.

The case for the ductility measure at n = 1 in (34) is completely unacceptable and it is discarded. This leaves the n = 2 case of (34) as

$$D^* = \left(\nu^*\right)^2 \tag{46}$$

and as being the most likely measure of ductility in terms of the renormalized Poisson's ratio ν^*. This measure of ductility in uniaxial tension is still provisional at this juncture, pending verification with explicit materials comparisons. The comparisons will be conducted shortly.

The significant values for D^*, ν^*, and ν from (46) and (28) are shown in Table 1. The ductile brittle transition at $D^* = 1/2$ has already been discussed. Also shown are the values at $D^* = 1/4$.

For lack of a better term this is called a special case because it does appear to have some physical significance. Very few materials have Poisson's ratios of less than 1/5. The coefficient of the dilatational term in (24) is half way between its maximum and minimum values of 1 and 0 at this value of Poisson's ratio.

	Ductility $D^*=(\nu^*)^2$	ν^*	ν
Totally Brittle	0	0	0
Special Case	1/4	1/2	1/5
D/B Transition	1/2	0.71	0.31
	3/4	0.87	0.41
Perfectly Ductile	1	1	1/2

Table 1, Ductility D^*, (46)

It is possible to determine E^* and ν^* other than from experiments directly on E and ν. From (18) it follows that

$$E^* = 2\mu \tag{47}$$

and

$$\nu^* = 1 - \frac{2\mu}{3k} \tag{48}$$

In most cases it is probably far easier to use (11) and (15) directly. However, relation (47) does incisively show that E^* is proportional to the distortional source of resistance while in (48), ν^* involves the interplay between the two independent physical effects, distortional and dilatational resistances to deformation. Relations (47) and (48) should be compared with the complex classical forms relating these four properties.

The comparison of ductility levels predicted by (46) with actual materials is given next as a critical test of the result (46). Table 2 shows a list of the solids forming common elements from the Periodic Table. The Poisson's ratios are from the sources referenced in [1]. The prediction of ductility levels in the last column of Table 2 is in accordance with general understanding of the spread of brittle to ductile behaviors for the different materials types. The ductile brittle transition of (43) and (45) occurs at the elements of nickel and cobalt which also is in reasonable accord with practice. This ductile brittle transition is not a transition in the thermodynamic sense of the term. But it is a meaningful transition as the division between predominately brittle versus predominately ductile groupings.

Element	ν	Ductility $D^*=(\nu^*)^2$
Absolute Limit, Perfect Ductility	1/2	1
Gold	0.44	0.84
Lead	0.44	0.84
Niobium	0.40	0.73
Palladium	0.39	0.71
Platinum	0.38	0.68
Silver	0.37	0.66
Vanadium	0.37	0.66
Tin	0.36	0.63
Aluminum	0.35	0.60
Copper	0.34	0.58
Tantalum	0.34	0.58
Titanium	0.32	0.53
Nickel	0.31	0.50
Cobalt	0.31	0.50
Magnesium	0.29	0.45
Iron	0.29	0.45
Tungsten	0.28	0.43
Zinc	0.25	0.36
Manganese	0.23	0.31
Uranium	0.23	0.31
Silicon	0.22	0.29
Plutonium	0.21	0.27
Chromium	0.21	0.27
Carbon (Diamond)	0.20	0.25
Limit For Most Elements/Materials	1/5	1/4
Beryllium	0.032	0.01
Absolute Limit, Total Brittleness	0	0

Table 2, Ductility of the solids forming elements

Even though Poisson's ratio is difficult to determine with high accuracy, the overall results of Table 2 establish the validity of using renormalized Poisson's ratio in (46) as a characterization of the ductility in uniaxial tension. It should be noted that the results in Table 2 are identical to the results found from a nanoscale analysis in [1].

The nanoscale analysis just mentioned was based upon differentiating the effects of bond stretching versus bond bending. The ductile end of the scale has no bond bending resistance compared with the bond stretching resistance. The brittle end of the scale maximizes the bond bending resistance relative to the bond stretching resistance. The present analysis has the ductile end of the scale as having no dilatational energy in (35) at $\nu^* = 1$. The brittle end of the scale has the dilatational energy in (35) as maximized at $\nu^* = 0$. These macroscopic scale explanations and understandings of behavior are completely compatible with the nanoscale explanations and understandings.

The nanoscale analysis in [1] also served the important purpose of proving that the physically unrealistic range of bond bending relative to bond stretching resistance disallows negative values of Poisson's ratio. Finally, it must be emphasized that the relation between Poisson's ratio and ductility has notable exceptions and its use here is mainly for studying fundamental effects rather than for general utility. In such exceptions, other physical effects are tailored to intercede, as with steel versus iron.

When interests are in a comprehensive treatment of ductile brittle matters, the account in Ref. [1] should be followed where the failure number, Fn, is derived and defined to specify the ductility level for any state of stress. The purpose here was entirely different. It was to establish the foundation of the new failure theory by carefully examining the connection between elasticity

theory and failure theory. The ductility characterization for uniaxial tension was shown to be the strong and supporting linkage between the two theories. From a failure point of view, it would be appropriate to say that this new development provided not just the coupling between the two theories but the integration into and unification of failure theory with elasticity theory.

Failure Mechanics Relationships and Significance

The term failure mechanics is here taken to cover and embrace failure theory for homogeneous and isotropic materials with all of its associations and all its implications and applications. It is the latest and perhaps the last in a long line of mechanics related and formulated disciplines.

It all began with the monumental and almost unthinkably difficult creation of classical mechanics by Isaac Newton. Then over about the following two centuries the disciplines of solid mechanics and fluid mechanics came into being and useful forms gradually evolved through the dedicated and inspired efforts of such great scientists as Cauchy , Bernoulli(s), Euler, Navier, Lame′, Poisson, Stokes and many, many others.

The dominant mechanics achievements of the 20^{th} century were the two disciplines of quantum mechanics and fracture mechanics. It is pointless to try to directly compare them because one is at the atomic scale and the other at the macroscopic scale. Both became vitally important.

In quantum mechanics the problem was to understand the behavior of the electrons orbiting the atomic nucleus under their opposite charges. Were the electrons best viewed as being particles or as waves? The evidence was contradictory. Then the possibility emerged that the behaviors were not deterministic but rather must be seen as probabilistic. That possibility was then

enlarged to more general concepts of the uncertainty of predictability. The complexity of the behaviors were enfolded and enshrouded in the full and glorious complications of three-dimensional interactions.

Over a prolonged and frenzied period of activities in the 1920's Bohr and Heisenberg and Dirac and Schrodinger and Born and others sorted through, debated and argued all the possibilities. The community ultimately reached a conclusion, something of an amalgam of all of the above. The almost unsolvable complexity of the quantum mechanics problem was at least partially due to its three dimensional nature. It was almost but not quite unsolvable. There finally was a workable resolution to the problem and it has carried forward and endured.

Fracture mechanics, like quantum mechanics, evolved and progressed through various stages in its development. Griffith [2] originated it and conceived the crack growth problem as an energy balance between the release of elastic strain energy against the work to create new crack surface. Irwin [3] had the insight to characterize the failure inception as related to the nature of the elastic singularity at the crack tip. This offered a normalizing and comparative capability. Rice [4] effected a further breakthrough by relating crack tip conditions to the properties of an invariant contour integral enclosing the crack. Barenblatt [5] re-examined crack tip conditions and found the non-singular stress state that helped to explain much of the behavior.

All of these efforts in combination formed the discipline of fracture mechanics. It has become enormously successful in design applications where materials are made into load bearing structures containing a single worst flaw (or flaws) that could be idealized as a crack (or cracks). Although fracture mechanics can be formulated in some special three-dimensional cases, its fundamental and wide ranging applicability is of a two-

dimensional nature and type. The success of fracture mechanics proved that the general subject of failure characterization no longer need be relegated to the state of just being an undisciplined agglomeration of empirical formulas.

Perhaps the final and completing mechanics related discipline will be that of the failure mechanics needed to characterize the three-dimensional failure behavior of homogeneous and isotropic materials. The fact that it is coming into existence only now was not for lack of effort over the years and over the centuries. Its great difficulty of development comes from the same source as that for quantum mechanics, namely its fully three dimensional nature having extremely complex interactions.

It is highly unlikely that failure mechanics, as envisioned here, could have been developed before the advent of fracture mechanics. Part of this is no doubt due to the clean and clear two-dimensional nature of fracture mechanics compared with the much greater complexity of general failure in three dimensions. Also, for the failure of homogeneous materials, there is no guiding certainty as to the site of failure initiation, as occurs with the pre-existing, stress concentrating cracks in fracture mechanics. Fracture mechanics had to lead the way for progress on failure. So the emergence of failure mechanics at this latter time has a reasonable coordination with the order and sequence of development of the other mechanics related disciplines.

There is no need to summarize the derivation and capability of failure mechanics. That is comprehensively treated in Ref. [1] under the heading of failure theory. The special interpretation developed herein however merits particular attention. Elasticity theory and failure theory for homogeneous and isotropic materials are not absolutely independent constructs. One without the other is incomplete. Although there is no single parameter that appears in both theories and bridges between them, there nevertheless is a

more subtle but no less effective coordination or coupling between them.

To explain this coupling between the two theories it is important and even necessary to recognize that any failure theory that does not fully treat ductile versus brittle failure conditions is not complete. Even more serious, such an incomplete failure theory cannot be assessed and evaluated, its status must be uncertain, even of doubtful validity until proven otherwise. The failure theory developed in [1] has the strengths ratio T/C and the stress states as determining the ductile versus brittle behaviors. More specifically, for the most important special case of uniaxial tension, the ratio T/C itself directly delineates the degree of ductility over the range from 0 to 1.

In the case of linear elasticity theory, the work here has shown that renormalized Poisson's ratio, ν^*, also gives a direct measure of the degree of ductility for uniaxial tension. This occurs through (46) over the same range of 0 to 1 as that for T/C.

It is thus the ductility measures from both the failure theory and from elasticity theory that provides the bridge between the two theories. At first it is exceedingly surprising that elasticity theory could say anything at all about the ductility in failure. But on deeper reflection it is not so surprising. The atomic scale characteristics of bond stretching and bond bending directly determine and differentiate not only T and C but also ν. Poisson's ratio, ν, actually contains a great deal of information that has heretofore been ignored or vastly under-appreciated.

Poisson's ratio, ν, is not at all secondary to the elastic modulus, E, it is the full and equal partner to E and of great significance in its own right. This new and enlarging insight became fairly apparent when the renormalized theory of elasticity was obtained

with the renormalized Poisson's ratio. Contrast this with the classical form of elasticity theory, usually using the two Lame′ constants λ and μ. The λ property has no physical interpretation at all. E and ν or better yet their renormalized forms E^* and ν^* represent the most meaningful statements of elastic properties, even with direct implications for failure. In association with E^* and ν^*, the uniaxial strengths T and C turn out to give the most meaningful and useful calibrating anchors for failure theory.

It is the failure theory of Ref. [1] (and herein) in its broadest context that comprises the discipline of failure mechanics. The comparison of failure mechanics (for solids) with fluid mechanics and with quantum mechanics is both illuminating and technically useful. In fluid mechanics and for slow flows, the Navier-Stokes equations give a linear behavior between the shear stress and the velocity gradient. But as the velocities increase, the nonlinear effect of the inertial source invalidates the previous linear relations and leads to turbulence.

For failure mechanics a similar situation evolves. Small deformation states produce linear relationships among the field variables of elasticity theory. But the nonlinear intercession of failure theory invalidates the previous conservative, linear behaviors. Nonlinear intercessions provide the limits to the linear behaviors of both solids and fluids. Elasticity theory without failure theory is open ended and incomplete. They must go together to provide closure.

The comparison between failure mechanics and quantum mechanics of course does not contain any analogies or congruencies between them. But both do share an overriding common characteristic that had much to do with their histories of development. Perhaps this would be better described as the common barriers that greatly impeded their respective

developments. The commonality is simply this. Neither of these two mechanics disciplines yielded to the standard and ordinary expectations. Both solutions turned out to be strongly counter-intuitive and it took major expenditures of time and effort to ultimately synthesize them, one over a highly intensive time period and the other over an agonizingly long and drawn out time extension.

It was well worth the extreme efforts in order to gain both disciplines. More broadly, all six of these mechanics related disciplines: classical mechanics, solid mechanics, fluid mechanics, quantum mechanics, fracture mechanics, and failure mechanics, comprise irreplaceable repositories of the most basic scientific/mechanistic knowledge.

References

1. Christensen, R. M. (2013), The Theory of Materials Failure, Oxford University Press, Oxford, U. K.

2. Griffith, A. A. (1921), "The Phenomena of Rupture and Flow in Solids", Trans. Royal Society of London, A221, 163-197.

3. Irwin, G. R. (1957), Analysis of Stresses and Strains Near the End of a Crack Traversing a Plate," Journal of Applied Mechanics, 24, 361-364.

4. Rice, J. R. (1968), "A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks," Journal of Applied Mechanics, 35, 379-386.

5. Barenblatt, G. I. (1962), "The Mathematical Theory of Equilibrium of Cracks in Brittle Fracture," Advances in Applied Mechanics, 7, 55-129.

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